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On three-body scattering cross sections†

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Abstract. Using a Hilbert space version of the Faddeev method, we prove finiteness and continuity as a function of the energy of the total scattering cross section for three quantum mechanical particles with two-body initial states. We assume that the pair potentials are locally in $L^q(\mathbb{R}^3)$ for some $q>\frac{3}{2}$ and decay as $|\boldsymbol{x}|^{-2-\delta}$ at infinity.

1. Introduction

It is well known that the quantum mechanical scattering problem for a non-relativistic particle by a local potential V leads to a finite total scattering cross section if $V(\boldsymbol{x})$ tends to zero faster than $|\boldsymbol{x}|^{-2-\delta}$ as $|\boldsymbol{x}|\to\infty$ for some $\delta>0$, whereas a decrease as $|\boldsymbol{x}|^{-2}$ or more slowly than $|\boldsymbol{x}|^{-2}$ gives rise to an infinite total cross section in general (for a more exact borderline, involving logarithmic factors, see Martin (1979)). For the three-body problem with local pair potentials $V_{ij}(\boldsymbol{r}_i-\boldsymbol{r}_j)$, one again expects that the total scattering cross section for a two-cluster initial channel should be finite if all $V_{ij}(\boldsymbol{x})$ decay faster than $|\boldsymbol{x}|^{-2-\delta}$ as $|\boldsymbol{x}|\to\infty$. It is the purpose of the present paper to prove this result under a suitable assumption on the local singularities of V_{ij}.

The finiteness of the total scattering cross section, integrated over a range of energies, for N-body systems with pair potentials decaying like $|\boldsymbol{x}|^{-2-\delta}$, has recently been proven by time-dependent methods (Amrein *et al* 1979, Enss and Simon 1980). These methods, though mathematically rather simple, are not suitable for making statements about the scattering cross section at fixed values of the energy. To obtain results at fixed energy, it still seems necessary to use the more elaborate stationary method, which leads to exact expressions for the T matrix. For the three-body problem, the essence of this approach is some kind of Faddeev equation. The original work of Faddeev (1965) uses hypotheses on the Fourier transforms of the pair potentials V_{ij}, which essentially require that $V_{ij}(\boldsymbol{x})\to 0$ at infinity faster than $|\boldsymbol{x}|^{-3-\delta}$. The hypotheses of Faddeev do imply the finiteness of the scattering amplitude at all scattering angles.

Faddeev's method has been formulated in Hilbert space language by Ginibre and Moulin (1974) (see Newton 1971, Thomas 1975, Howland 1976, Mourre 1977 for related work), and in this form it suffices to make a hypothesis on the pair potentials themselves rather than on their Fourier transforms. Ginibre and Moulin prove asymptotic completeness if

$$V_{ij}(\boldsymbol{x})=V_{1,ij}(\boldsymbol{x})+V_{2,ij}(\boldsymbol{x})=(1+|\boldsymbol{x}|)^{-2-\delta}(W_{1,ij}(\boldsymbol{x})+W_{2,ij}(\boldsymbol{x}))\qquad(1)$$

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with $\delta > 0$, $W_{1,ij} \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $p > \frac{3}{2} > q$ and $W_{2,ij} \in L^\infty(\mathbb{R}^3)$. Amrein *et al* (1977) have used this Hilbert space approach to prove the finiteness of the total scattering cross section for bounded potentials decaying as $|\boldsymbol{x}|^{-3-\delta}$. The extension of their method to the class of potentials given by (1) involves some technicalities which will be described in the sequel, the principal one being a choice different from that made in earlier publications for the auxiliary operator C_a introduced in lemma 1.

Section 2 contains the definition of the various Hamiltonians used in the three-body problem. In § 3 we explain the Faddeev method in Hilbert space and its relation to the S matrix, and in § 4 we establish the finiteness of the total scattering cross section and its continuity as a function of the energy. A certain number of auxiliary results will be announced in the form of lemmas, the proofs of which will all be collected in the appendix.

We shall use some of the results of chapter 16 of Amrein *et al* (1977). This reference may also be consulted for a motivation of the Faddeev method in Hilbert space and for additional details and will be cited as AJS. The following notations will be used: $D(A)$ denotes the domain of the linear operator A, $\mathscr{B}(\mathscr{H}, \mathscr{H}')$ or $\mathscr{B}(\mathscr{H}) \equiv \mathscr{B}(\mathscr{H}, \mathscr{H})$ the set of all bounded everywhere defined linear operators from the Hilbert space $\mathscr{H}$ to $\mathscr{H}'$ or $\mathscr{H}$ respectively, $\mathscr{B}_2(\mathscr{H}, \mathscr{H}')$ the set of all Hilbert–Schmidt operators from $\mathscr{H}$ to $\mathscr{H}'$, $\mathscr{B}_4(\mathscr{H}, \mathscr{H}')$ the set of all operators A in $\mathscr{B}(\mathscr{H}, \mathscr{H}')$ such that $A^*A \in \mathscr{B}_2(\mathscr{H})$, and $\mathscr{B}_\infty(\mathscr{H})$ the set of all compact operators in $\mathscr{H}$. $\|A\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm of A, $\|A\|_4 \equiv \|A^*A\|_{\mathrm{HS}}^{1/2}$ its $\mathscr{B}_4$ norm and $\|f\|$ the L^2 norm of the function f defined on $\mathbb{R}^3$ or $\mathbb{R}^6$. Throughout the paper, the letter δ refers to the number appearing in (1). We set $W_{ij} = W_{1,ij} + W_{2,ij}$ and let $|W_{ij}|^{1/2}$ and $W_{ij}^{1/2}$ be the multiplication operators by $|W_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)|^{1/2}$ and

$$|W_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)|^{1/2} \operatorname{sgn} W_{ij}(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

respectively.

2. Hamiltonian operators in the three-body problem

We consider three spinless particles and denote by m_j the mass of the jth particle and by $\boldsymbol{r}_j \in \mathbb{R}^3$ its position vector. We use the letters a, b, c, d to label pairs of particles, i.e. $a = \{1, 2\}, \{1, 3\}$ or $\{2, 3\}$. For each value of a, we introduce two relative coordinates $\boldsymbol{x}_a$ and $\boldsymbol{y}_a$ as follows: if $a = \{k, l\}$, then $\boldsymbol{x}_a = \boldsymbol{r}_l - \boldsymbol{r}_k$ and

$$\boldsymbol{y}_a = \boldsymbol{r}_{kl} - (m_k + m_l)^{-1}(m_k \boldsymbol{r}_k + m_l \boldsymbol{r}_l),$$

where $\boldsymbol{r}_{kl}$ is the position vector of the particle not included in the pair $\{k, l\}$. If, for example, $\{k, l\} = \{1, 2\}$, then $|\boldsymbol{x}_a|$ is the distance from particle 1 to particle 2, and $|\boldsymbol{y}_a|$ is the distance from the centre of mass of particles 1 and 2 to particle 3. We denote by $\boldsymbol{P}_a$ and $\boldsymbol{K}_a$ the relative momentum operators associated with the position variables $\boldsymbol{x}_a$ and $\boldsymbol{y}_a$ respectively, i.e. $\boldsymbol{P}_a = -\mathrm{i}\boldsymbol{\nabla}_{x_a}$ and $\boldsymbol{K}_a = -\mathrm{i}\boldsymbol{\nabla}_{y_a}$. We shall often use the fact that any three different relative coordinates are linearly dependent. For example we have $\boldsymbol{x}_a = \mu \boldsymbol{x}_b + \nu \boldsymbol{y}_b$, where μ and ν are constants (depending on a and b) that are both non-zero if $a \neq b$, or $\boldsymbol{x}_a = \mu' \boldsymbol{x}_b + \nu' \boldsymbol{x}_c$ if $b \neq c$, where μ' and ν' are again non-zero if a, b and c are all different.

After separating off the centre-of-mass motion of the three-particle system, its relative motion is described in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^6)$, where the six variables may be any of the three sets $\{\boldsymbol{x}_a, \boldsymbol{y}_a\}$. For each value of a, $L^2(\mathbb{R}^6)$ may be viewed as a tensor product $L^2(\mathbb{R}^6) = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$, where the variables in $\mathbb{R}^3$ are $\boldsymbol{x}_a$ and $\boldsymbol{y}_a$ respectively. To specify the choice of variables, we shall write this tensor product as $L^2_{x_a}(\mathbb{R}^3) \otimes L^2_{y_a}(\mathbb{R}^3)$.

For each value of a, the free Hamiltonian H_0 has the form

$$H_0 = \gamma_a \boldsymbol{P}_a^2 + \eta_a \boldsymbol{K}_a^2 \tag{2}$$

where γ_a and η_a are positive constants depending on the masses m_1, m_2 and m_3 and on a, i.e. on the choice of a set of relative coordinates. The total Hamiltonian is obtained by adding to H_0 the sum of three pair potentials

$$H = H_0 + \sum_a V_a \tag{3}$$

where V_a denotes the multiplication operator in $L^2(\mathbb{R}^6)$ by the real-valued function $V_a(\boldsymbol{x}_a)$. Under the hypothesis (1) on the pair potentials, ΣV_a is a small form perturbation of H_0 (this follows from the estimates in the proof of lemma 5(i)). Hence the self-adjoint operator H can be defined by adding the quadratic forms associated with H_0 and ΣV_a (see Faris 1975, Simon 1971 for details). This means in particular that $D(|H|^{1/2}) = D(H_0^{1/2}) \subseteq D(|V_a|^{1/2})$.

In addition to H_0 and H, we shall use the cluster Hamiltonians H_a which are defined as the form sum of H_0 and V_a, i.e. formally

$$H_a = (\gamma_a \boldsymbol{P}_a^2 + V_a) + \eta_a \boldsymbol{K}_a^2. \tag{4}$$

The first operator on the right-hand side acts only in the variable $\boldsymbol{x}_a$, the second one only in the variable $\boldsymbol{y}_a$. When viewed as an operator in $L^2_{x_a}(\mathbb{R}^3)$, the first operator $h_a = \gamma_a \boldsymbol{P}_a^2 + V_a$ represents the Hamiltonian for the relative motion of the two particles forming the pair a, interacting via the potential $V_a(\boldsymbol{x}_a)$. If $\mathcal{H}_p(h_a)$ denotes the subspace of $L^2_{x_a}(\mathbb{R}^3)$ spanned by the set of all eigenvectors of h_a, we fix an orthonormal basis $\{e_j^a\}$ of $\mathcal{H}_p(h_a)$ formed of eigenvectors of $h_a (j = 1, \ldots, n_a)$. Since there are three such sets of eigenvectors (as a varies), it is useful to label them by a single index α ($\alpha = 1, \ldots, n_{\{12\}} + n_{\{13\}} + n_{\{23\}}$). The notation $\alpha \div a$ will mean that the vector e_α associated with the number α is an eigenvector of the Hamiltonian h_a (i.e. one of the chosen vectors e_j^a). We denote the associated eigenvalue by λ_α: $h_a e_\alpha = \lambda_\alpha e_\alpha$.

For each α, we define the channel subspace $\mathcal{M}_\alpha$ in $L^2(\mathbb{R}^6)$ by

$$\mathcal{M}_\alpha := e_\alpha \otimes L^2_{y_a}(\mathbb{R}^3) \tag{5}$$

where $\alpha \div a$ and e_α is viewed as a function of $\boldsymbol{x}_a$. The direct sum of all $\mathcal{M}_\alpha$ with $\alpha \div a$ is called the cluster subspace $\mathcal{M}_a$:

$$\mathcal{M}_a := \bigoplus_{\alpha \div a} \mathcal{M}_\alpha. \tag{6}$$

The orthogonal projection with range $\mathcal{M}_\alpha$ or $\mathcal{M}_a$ will be denoted by E_α or E_a respectively. If $\alpha \div a$, H_a commutes with E_α and satisfies the relation

$$H_a E_\alpha = (\lambda_\alpha + \eta_a \boldsymbol{K}_a^2) E_\alpha. \tag{7}$$

In order to be able to control the resolvent $(H - z)^{-1}$ of the total Hamiltonian when the complex number z approaches the real axis, one has to know the spectral properties of the three two-body Hamiltonians h_a in sufficient detail. To formulate the assumptions on the spectrum of h_a, we factorise the potential V_a into $V_a = A_a B_a$, where A_a and B_a are the multiplication operators in $L^2(\mathbb{R}^6)$ by $A_a(\boldsymbol{x}_a) := |V_a(\boldsymbol{x}_a)|^{1/2}$

and $B_a(\boldsymbol{x}_a) := |V_a(\boldsymbol{x}_a)|^{1/2} \operatorname{sgn} V_a(\boldsymbol{x}_a)$ respectively. We denote by a_a and b_a the corresponding multiplication operators in $L^2_{x_a}(\mathbb{R}^3)$. Under the hypothesis (1), the operators

$$w^a_{\lambda \pm i0} := \lim_{\varepsilon \to +0} b_a(\gamma_a \boldsymbol{P}_a^2 + \lambda \mp i\varepsilon)^{-1} a_a \tag{8}$$

are compact operators in $L^2_{x_a}(\mathbb{R}^3)$. The spectral assumptions on h_a may be conveniently formulated in terms of $w^a_{\lambda \pm i0}$.

We assume that (S1) for each $\lambda \geqslant 0$, $w^a_{\lambda \pm i0}$ do not have the eigenvalue -1, (S2) the point $z = -1$ is an eigenvalue of $w^a_{\lambda \pm i0}$ for at most a finite number of (negative) λ.

In terms of h_a, (S1) and (S2) mean that the spectrum of h_a on $[0, \infty)$ is purely absolutely continuous, that there is no zero-energy resonance and that h_a has only a finite number of (strictly negative) eigenvalues, all of which are of finite multiplicity (AJS ch 10). We shall also need to know a weak decay property of the eigenfunctions of h_a, namely that (S3)

$$(1 + |\boldsymbol{x}_a|)^{2+\delta} e_\alpha(\boldsymbol{x}_a) \in L^2(\mathbb{R}^3)$$

where δ is the number appearing in (1). A discussion of (S1)–(S3) for the class of potentials satisfying (1) is given in remark 4. The following consequence of (S3) will be important.

Lemma 1. Assume that V_a is of the form (1) and let $e_\alpha \in L^2_{x_a}(\mathbb{R}^3)$ be an eigenvector of h_a satisfying (S3). Then $(1 + |\boldsymbol{x}_a|)^\nu e_\alpha(\boldsymbol{x}_a) \in D(|\boldsymbol{P}_a|)$ for each $\nu \in [0, 2 + \delta)$, where $|\boldsymbol{P}_a| := (\boldsymbol{P}_a^2)^{1/2}$ is viewed as an operator in $L^2_{x_a}(\mathbb{R}^3)$.

This property of the eigenvectors of h_a will allow us to prove the following results.

Lemma 2. Let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be defined as $\rho(\boldsymbol{x}) = (1 + |\boldsymbol{x}|)^{-\frac{1}{2} - \delta/4}$, and denote by C_a the multiplication operator in $L^2(\mathbb{R}^6)$ by $\rho(\boldsymbol{y}_a)$. Then $C_a E_a = E_a C_a$. If a and b are arbitrary and $c \neq d$, each of the following operators is in $\mathcal{B}(L^2(\mathbb{R}^6))$:

(i) $\quad |W_a|^{1/2}(|\boldsymbol{P}_b| + 1)^{-1} \qquad B_a(|\boldsymbol{P}_b| + 1)^{-1} \qquad |W_a|^{1/2} E_b \qquad B_a E_b$

(ii) $\quad C_a^{-1} C_b^{-1} A_d E_c \qquad C_c^{-2} E_c A_d \qquad C_c^{-1} E_c C_d^{-1} A_d$

(iii) $\quad C_a^{-1} E_a C_b.$

Each eigenvalue λ_α of h_a forms a scattering threshold for the three-body problem. We define $\Theta_a := \{\lambda_\alpha \mid \alpha \div a\}$ and $\Theta := \cup_a \Theta_a \cup \{0\}$. θ consists of $n_{\{12\}} + n_{\{13\}} + n_{\{23\}} + 1$ (not necessarily different) numbers, namely the $n_{\{12\}} + n_{\{13\}} + n_{\{23\}}$ negative thresholds for the two-body scattering channels introduced above, and the point $\lambda_0 = 0$ which is the threshold for the three-body channel corresponding to three freely moving particles. We shall set $\alpha = 0$ when referring to this latter channel. The corresponding channel subspace is $\mathcal{M}_0 = L^2(\mathbb{R}^6)$, i.e. the entire Hilbert space, and the corresponding channel Hamiltonian is the operator H_0 defined in (2).

3. The three-body resolvent and the scattering amplitude

For the class of potentials (1), the existence of the wave operators for the three-body problem, the asymptotic completeness of the scattering theory and hence the existence of a family of scattering operators $S_{\beta\alpha}$ satisfying the unitarity relation were established by Ginibre and Moulin (1974). $S_{\beta\alpha}$ maps $\mathcal{M}_\alpha$ into $\mathcal{M}_\beta$ and is the scattering operator for scattering from the initial channel α into the final channel β. One usually considers the case where the initial channel is a two-body channel, whereas the final state may have a component in each channel β, including the three-body channel $\beta = 0$.

If $\alpha \div a$, the scattering amplitude from channel α to channel β at energy λ is formally given as

$$f_{\beta\alpha}(\lambda; \boldsymbol{\omega}_\alpha \to \boldsymbol{\omega}_\beta) = \lim_{\varepsilon \to +0} c_{\beta\alpha}(\lambda)\left\langle \lambda, \boldsymbol{\omega}_\beta; \beta \left| \sum_{d \neq a} V_d - \sum_c{}' \sum_{d \neq a} V_c(H - \lambda - \mathrm{i}\varepsilon)^{-1} V_d \right| \lambda, \boldsymbol{\omega}_\alpha; \alpha \right\rangle \tag{9}$$

where we have set $\hbar = 1$. Here $|\lambda, \boldsymbol{\omega}_\beta; \beta\rangle$ is an improper eigenstate of the channel Hamiltonian $H_\beta = \lambda_\beta + \eta_b \boldsymbol{K}_b^2$ $(\beta \div b)$, viewed as an operator in $\mathcal{M}_\beta$. This eigenstate is determined by the value λ of the energy and by another variable $\boldsymbol{\omega}_\beta$. If $\beta \neq 0$, $\boldsymbol{\omega}_\beta$ is a vector on the unit sphere S^2 in $\mathbb{R}^3$, which describes the direction of the relative momentum of the two-body cluster and the free particle in channel β. If $\beta = 0$, $\boldsymbol{\omega}_\beta$ varies over a five-dimensional ellipsoid $\mathscr{E}^5 := \{(\boldsymbol{p}_c, \boldsymbol{k}_c) \mid \gamma_c \boldsymbol{p}_c^2 + \eta_c \boldsymbol{k}_c^2 = 1\}$. We shall write $\boldsymbol{\omega}_\beta = (\boldsymbol{\omega}, \boldsymbol{q})$, where $\boldsymbol{\omega} = \boldsymbol{p}_c/|\boldsymbol{p}_c|$ and $\boldsymbol{q} = (\eta_c/\lambda)^{1/2}\boldsymbol{k}_c$ and c is any of the three pair indices. $\boldsymbol{\omega}$ gives the direction of the relative momentum of the two particles in the pair c and $(\lambda/\eta_c)^{1/2}\boldsymbol{q}$ the relative momentum of the third particle with respect to the centre of mass of this pair. The constants $c_{\beta\alpha}(\lambda)$ in (9) depend on the normalisation of the states $|\lambda, \boldsymbol{\omega}_\gamma; \gamma\rangle$ and will not be needed explicitly. The sum Σ_c' runs over all values $c \neq b$ when $0 \neq \beta \div b$ and over all three values $c = \{1, 2\}, \{1, 3\}, \{2, 3\}$ when $\beta = 0$.

We are concerned with the finiteness and continuity in λ of the total scattering cross section, i.e. with the finiteness and continuity in λ of $\int |f_{\beta\alpha}(\lambda; \boldsymbol{\omega}_\alpha \to \boldsymbol{\omega}_\beta)|^2 \, \mathrm{d}\omega_\beta$. More precisely, we shall discuss the average of this quantity over the initial direction $\boldsymbol{\omega}_\alpha$, i.e.

$$\bar{\sigma}_{\beta\alpha}(\lambda) := \frac{1}{4\pi} \int \mathrm{d}\omega_\alpha \, \mathrm{d}\omega_\beta \, |f_{\beta\alpha}(\lambda; \boldsymbol{\omega}_\alpha \to \omega_\beta)|^2. \tag{10}$$

If $\bar{\sigma}_{\beta\alpha}(\lambda)$ is finite for each final channel β, then the averaged total cross section

$$\bar{\sigma}_{\alpha,\mathrm{tot}} := \sum_\beta \bar{\sigma}_{\beta\alpha}(\lambda) \tag{11}$$

for scattering initiated in channel α will also be finite, since the number of scattering channels is finite by (S2). The energy λ in (11) lies in the interval (λ_α, ∞), and the sum is effectively only over those channels β that are open at the energy λ, i.e. such that $\lambda_\beta \leq \lambda$.

In order to estimate $\bar{\sigma}_{\beta\alpha}(\lambda)$, one has to give a precise meaning to the formal expression (9). This involves the following two problems: (i) study the limit of $V_c(H - \lambda - \mathrm{i}\varepsilon)^{-1} V_d$ as $\varepsilon \to 0$ (notice that $(H - \lambda - \mathrm{i}\varepsilon)^{-1}$ becomes unbounded when $\varepsilon \to 0$), (ii) give a precise meaning to the 'eigenstates' $|\lambda, \boldsymbol{\omega}_\beta; \beta\rangle$. Problem (i) is solved by using a variant of the Faddeev method to show that certain auxiliary operators related to $B_c(H - \lambda - \mathrm{i}0)^{-1} C_d$ are bounded and depend continuously on λ, and problem (ii) is handled by using the spectral representation of the self-adjoint channel Hamiltonians H_β.

To study the operators $Y_z^{cd} := B_c(H - z)^{-1} C_d$ in the neighbourhood of the real axis, it is convenient to introduce the Hilbert space $\mathscr{K} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ and to attach the label $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ to the first, second and third summand respectively. The family of nine operators Y_z^{cd} (each of which acts in $\mathscr{H} = L^2(\mathbb{R}^6)$) can then be combined into a single operator Y_z acting in $\mathscr{K}$; Y_z is an operator-valued 3×3 matrix with entries Y_z^{cd}. We shall write $Y_z=\{Y_z^{cd}\}$. The Faddeev method allows one to study Y_z in terms of the simpler operator $\{B_cR_z^cC_d\}$, where $R_z^c:=(H_c-z)^{-1}$ is the resolvent of the cluster Hamiltonian H_c. In this method it is essential to treat separately the part of R_z^c corresponding to the bound states of the two-body Hamiltonian h_c (i.e. $R_z^cE_c$) and the part of R_z^c corresponding to the continuous spectrum of h_c (i.e. $R_z^c(I-E_c)$). For this purpose we introduce yet another Hilbert space $\hat{\mathscr{K}}=\mathscr{K}\oplus\mathscr{K}$. We label objects pertaining to the first summand by an index 0 and objects pertaining to the second one by an index 1, and we shall use these two summands to treat $R_z^c(I-E_c)$ and $R_z^cE_c$ respectively. An operator $\hat{A}$ acting in $\hat{\mathscr{K}}$ may be viewed as a 2×2 matrix $\{A_{ik}\}_{i,k=0}^1$, where the entries A_{ik} are operators acting in $\mathscr{K}$. $\hat{I}$ will be the identity operator in $\hat{\mathscr{K}}$.

For each non-real z we define an operator K_z from $\mathscr{K}$ to $\hat{\mathscr{K}}$ and an operator J_z from $\hat{\mathscr{K}}$ to $\mathscr{K}$ as follows

$$K_zg=K_z^0g\oplus K_z^1g \qquad J_z(f_0\oplus f_1)=f_0+G_zf_1 \tag{12}$$

where $f_0, f_1, g\in\mathscr{K}$ and K_z^0, K_z^1 and G_z are operators in $\mathscr{K}$ given by

$$K_z^0=\{B_c(I-E_c)R_z^cC_d\} \qquad K_z^1=\{C_c^{-1}E_cC_d\} \tag{13}$$

$$G_z=\{B_cE_cR_z^cC_c\delta_{cd}\}. \tag{14}$$

We also introduce the operator $\hat{D}_z=\{D_{z,ik}\}$ acting in $\hat{\mathscr{K}}$ by

$$D_{z,00}=\{B_c(I-E_c)R_z^cA_d(1-\delta_{cd})\} \qquad D_{z,01}=D_{z,00}G_z \tag{15}$$

$$D_{z,10}=\{C_c^{-1}E_cA_d(1-\delta_{cd})\} \qquad D_{z,11}=D_{z,10}G_z. \tag{16}$$

We notice that K_z^1 and $D_{z,10}$ do not depend on z and are bounded operators by lemma 2. The boundedness of the other operators defined in (13)–(16) and of Y_z essentially follows from the definition of H and H_c and will be verified in lemma 5 in the appendix.

By simple algebraic manipulations, using the second resolvent equation and lemma 5, one finds that these operators have the following properties (AJS, Propositions 16.5, 16.7(a) and equation (16.20)):

$$\hat{I}+\hat{D}_z \text{ is an invertible operator in } \hat{\mathscr{K}}, \text{ and } (\hat{I}+\hat{D}_z)^{-1}\in\mathscr{B}(\hat{\mathscr{K}}) \tag{17}$$

$$Y_z=J_z(\hat{I}+\hat{D}_z)^{-1}K_z. \tag{18}$$

We notice that $\hat{D}_z$ has a special structure, namely that it may be written as $\hat{D}_z=\hat{N}+(\hat{D}_z-\hat{N})$, where

$$\hat{N}:=\begin{pmatrix}0 & 0\\ D_{z,10} & 0\end{pmatrix}$$

does not depend on z and is nilpotent: $\hat{N}^2=0$. Setting $\hat{\mathscr{W}}_z=(\hat{I}-\hat{N})(\hat{D}_z-\hat{N})$, we find from this that

$$\hat{I}+\hat{D}_z=(\hat{I}+\hat{N})(\hat{I}+\hat{\mathscr{W}}_z). \tag{19}$$

Since $(\hat{I}-\hat{N})(\hat{I}+\hat{N})=\hat{I}=(\hat{I}+\hat{N})(\hat{I}-\hat{N})$, we have $(\hat{I}+\hat{N})^{-1}=(\hat{I}-\hat{N})$. Hence, by (17) above, $(\hat{I}+\hat{\mathscr{W}}_z)\equiv(\hat{I}-\hat{N})(\hat{I}+\hat{D}_z)$ is invertible and we have

$$(\hat{I}+\hat{D}_z)^{-1}=(\hat{I}+\hat{\mathscr{W}}_z)^{-1}(\hat{I}-\hat{N}). \tag{20}$$

In the next lemma we collect the properties of the operators $\hat{\mathscr{W}}_z$ and then explain how these are related to the boundary values of Y_z on the real axis.

Lemma 3. (i) For each non-real z, $\hat{\mathscr{W}}_z$ is a compact operator in $\hat{\mathscr{H}}$.

(ii) For each $\lambda \in \mathbb{R}\backslash\Theta$, the limit $\hat{\mathscr{W}}_{\lambda+i0} \equiv \lim \hat{\mathscr{W}}_{\lambda+i\varepsilon}$ as $\varepsilon \to +0$ exists in the operator norm, and the convergence is uniform on each compact subset of $\mathbb{R}\backslash\Theta$.

We now define Γ_0 to be the set of all λ in $\mathbb{R}\backslash\Theta$ for which $\hat{I} + \hat{\mathscr{W}}_{\lambda+i0}$ is not invertible. By the results of lemma 2 and a local version of lemma 9.5 of AJS (see p 389), it follows that $\Gamma_0 \cup \Theta$ is a closed subset of $\mathbb{R}$ of Lebesgue measure zero and that $(\hat{I} + \hat{\mathscr{W}}_{\lambda+i\varepsilon})^{-1}$ is a norm-continuous function of λ and ε when λ varies over a compact subset Δ of $\mathbb{R}\backslash(\Gamma_0 \cup \Theta)$ and ε over $[0, 1]$. Γ_0 is called the exceptional set associated with the family $\{\hat{\mathscr{W}}_z\}_{\mathrm{Im}\, z>0}$ and will be briefly discussed in remark 4.

In view of the identity (20), the preceding statements lead to the following proposition.

Proposition 1. Let z be non-real or $z = \lambda + i0$ with $\lambda \notin \Gamma_0 \cup \Theta$. Then $\hat{I} + \hat{D}_z$ is invertible, and $(\hat{I} + \hat{D}_z)^{-1} \in \mathscr{B}(\hat{\mathscr{H}})$. If Δ is any compact subset of $\mathbb{R}\backslash(\Gamma_0 \cup \Theta)$ and $\Delta_+ := \{z \in \mathbb{C} \mid \mathrm{Re}\, z \in \Delta, \mathrm{Im}\, z \in [0, 1]\}$, then $(\hat{I} + \hat{D}_z)^{-1}$ is a norm-continuous function of z on Δ_+.

We now introduce the spectral representation of the channel Hamiltonian H_α. If $\alpha \neq 0$, this spectral representation is obtained by a unitary operator $\mathscr{U}_\alpha$ mapping the subspace $\mathscr{M}_\alpha$ of $L^2(\mathbb{R}^6)$ onto $\mathscr{G}_\alpha := L^2([\lambda_\alpha, \infty), L^2(S^2))$, the Hilbert space of all measurable functions defined on $[\lambda_\alpha, \infty)$ with values in $L^2(S^2)$ (the Hilbert space of square-integrable functions on the unit sphere S^2) such that

$$\int_{\lambda_\alpha}^{\infty} \|f_\lambda\|^2_{L^2(S^2)}\, d\lambda < \infty.$$

Here f_λ denotes the value of f at the point λ. The unitary operator $\mathscr{U}_\alpha$ is such that $\mathscr{U}_\alpha H_\alpha \mathscr{U}_\alpha^{-1}$ is the multiplication operator by λ in $\mathscr{G}_\alpha$ and is given by

$$(\mathscr{U}_\alpha f)_\lambda(\boldsymbol{\omega}_\alpha) = (2\pi)^{-3/2}\mu_\alpha^{1/2} k_\alpha^{1/2} \int \exp(-i k_\alpha \boldsymbol{\omega}_\alpha \cdot \boldsymbol{y}_a) f(\boldsymbol{y}_a)\, d^3 y_a \tag{21}$$

where $\alpha \div a$, $k_\alpha = [2\mu_a(\lambda - \lambda_\alpha)]^{1/2}$, μ_a is the reduced mass of the two clusters in channel α, and $f \in L^2_{y_a}(\mathbb{R}^3)$, i.e. $e_\alpha \otimes f \in \mathscr{M}_\alpha$. We also define the operator $U^\alpha : L^2(\mathbb{R}^6) \to L^2(S^2)$ by multiplying $\mathscr{U}_\alpha$ with E_α: $U^\alpha := \mathscr{U}_\alpha E_\alpha$.

For $\alpha = 0$, we consider three spectral representations of H_0 labelled by a pair index c. U_c^0 is a unitary operator from $L^2(\mathbb{R}^6)$ onto $\mathscr{G}_c := L^2([0, \infty), L^2(\mathscr{E}^5))$ diagonalising H_0, given by

$$(U_c^0 f)_\lambda(\boldsymbol{\omega}, \boldsymbol{q}) = (2\pi)^{-3} h_c(\lambda, q) \int \exp\left[-i\left(\frac{\lambda}{\gamma_c}(1-q^2)\right)^{1/2} \boldsymbol{\omega} \cdot \boldsymbol{x}_c - i\left(\frac{\lambda}{\eta_c}\right)^{1/2} \boldsymbol{q} \cdot \boldsymbol{y}_c\right] f(\boldsymbol{x}_c, \boldsymbol{y}_c)\, d^3 x_c\, d^3 y_c \tag{22}$$

where

$$h_c(\lambda, q) = \tfrac{1}{2}(2\pi\gamma_c\eta_c)^{-3/2}(1-q^2)^{1/2}\lambda^2 q^2 \qquad q \equiv |\boldsymbol{q}|. \tag{23}$$

The passage between different spectral representations of H_0 is given by the unitary operators $U_b^0 U_c^{0*}$ mapping $\mathscr{G}_c$ onto $\mathscr{G}_b$. These operators have the form $(U_b^0 U_c^{0*} f)_\lambda = U_{bc} f_\lambda$, where U_{bc} is a λ-independent unitary operator in $L^2(\mathscr{E}^5)$, induced by the coordinate transformation $(\boldsymbol{p}_c, \boldsymbol{k}_c) \mapsto (\boldsymbol{p}_b, \boldsymbol{k}_b)$.

We also consider a class of operators $M^{\beta}(A,\lambda)$ and $M_c^0(A,\lambda)$ mapping $L^2(\mathbb{R}^6)$ into $L^2(S^2)$ and into $L^2(\mathscr{E}^5)$ respectively. These operators depend on the energy λ and a bounded operator A acting in $L^2(\mathbb{R}^6)$. For $\beta \neq 0$, we set

$$[M^{\beta}(A,\lambda)f](\boldsymbol{\omega}_{\beta}) = (U^{\beta}Af)_{\lambda}(\boldsymbol{\omega}_{\beta}) \tag{24}$$

whereas

$$[M_c^0(A,\lambda)f](\boldsymbol{\omega},\boldsymbol{q}) = (U_c^0Af)_{\lambda}(\boldsymbol{\omega},\boldsymbol{q}). \tag{25}$$

$M_c^0(A,\lambda)f$ for example is obtained by restricting the Fourier transform of the vector Af to the ellipsoid $\{(\boldsymbol{p}_c,\boldsymbol{k}_c)\,|\,\gamma_c\boldsymbol{p}_c^2+\eta_c\boldsymbol{k}_c^2=\lambda\}$. The following properties of these operators will be used (χ_{Δ} denotes the characteristic function of the Borel set $\Delta \subseteq \mathbb{R}$, E_{Δ}^0 the spectral projection of H_0 associated with Δ):

$$M^{\beta}(AB,\lambda) = M^{\beta}(A,\lambda)B \tag{26}$$

$$M^{\beta}(A,\lambda) = M^{\beta}(E_{\beta}A,\lambda) \tag{27}$$

$$M_c^0(A,\lambda) = \chi_{\Delta}(\lambda)M_c^0(A,\lambda) = M_c^0(E_{\Delta}^0A,\lambda) \qquad \text{if } \lambda \in \Delta. \tag{28}$$

We notice that the adjoints $M^{\beta}(A,\lambda)^*$ and $M_c^0(A,\lambda)^*$ of these operators map $L^2(S^2)$ and $L^2(\mathscr{E}^5)$ respectively into $L^2(\mathbb{R}^6)$.

We are mostly interested in the case where A is a multiplication operator in $L^2(\mathbb{R}^6)$. In the next lemma we specify sufficient conditions on A implying that $M^{\beta}(A,\lambda)$ or $M_c^0(A,\lambda)$ are reasonable operators (for instance bounded or in the Hilbert–Schmidt class). If A is the multiplication operator by a function $\varphi\colon \mathbb{R}^6 \to \mathbb{C}$ or by a function $\psi(\boldsymbol{x}_a)$, we shall denote the associated operator $M^{\beta}(A,\lambda)$ by $M^{\beta}(\varphi,\lambda)$ and $M^{\beta}(\psi_a,\lambda)$ respectively.

Lemma 4. (i) Let $m=0,1,2$ or 3, $a \neq b$ and $|\varphi_m(\boldsymbol{x}_a,\boldsymbol{x}_b)| \leqslant \rho(\boldsymbol{x}_a)^{-m}\psi(\boldsymbol{x}_b)$ with $\psi \in L^2(\mathbb{R}^3)$, where ρ is as in lemma 2. Then, for $0 \neq \alpha \div a$, $M^{\alpha}(\varphi_m,\lambda) \in \mathscr{B}_2(L^2(\mathbb{R}^6), L^2(S^2))$ for all $\lambda \geqslant \lambda_{\alpha}$ and is a continuous function of λ in Hilbert–Schmidt norm.

(ii) For $0 \neq \alpha \div a \neq b$, $M^{\alpha}(A_b,\lambda)$ and $M^{\alpha}(C_a^2,\lambda)$ belong to $\mathscr{B}_4(L^2(\mathbb{R}^6), L^2(S^2))$ for all $\lambda \geqslant \lambda_{\alpha}$ and are continuous in $\mathscr{B}_4$ norm.

(iii) Let $\psi(\boldsymbol{x}_a) \in L^3(\mathbb{R}^3)$. Then $M_a^0(\psi_a,\lambda) \in \mathscr{B}(L^2(\mathbb{R}^6), L^2(\mathscr{E}^5))$ and is strongly continuous, for all $\lambda \geqslant 0$. $M_a^0(\psi_a,\lambda)$ is never compact if $\psi \neq 0$.

(iv) $M^{\alpha}(A,\lambda)$ and $M_a^0(A,\lambda)$ satisfy the following properties. If $g \in \mathscr{S}(\mathbb{R}^6)$, then $(\alpha \div a \neq b)$

$$B_b^*E_{\alpha}g = \int M^{\alpha}(B_b,\lambda)^*(U^{\alpha}g)_{\lambda}\,\mathrm{d}\lambda$$

$$\operatorname*{w\text{-}lim}_{\varepsilon\to+0} B_b^*E_{\alpha}(R_{\lambda+\mathrm{i}\varepsilon}^{\alpha}-R_{\lambda-\mathrm{i}\varepsilon}^{\alpha})g = 2\pi\mathrm{i}M^{\alpha}(B_b,\lambda)^*(U^{\alpha}g)_{\lambda}$$

$$\psi_a g = \int M_a^0(\psi_a,\lambda)^*(U_a^0g)_{\lambda}$$

and

$$\operatorname*{w\text{-}lim}_{\varepsilon\to+0} \psi_a(R_{\lambda+\mathrm{i}\varepsilon}^0-R_{\lambda-\mathrm{i}\varepsilon}^0)g = 2\pi\mathrm{i}M_a^0(\psi_a,\lambda)^*(U_a^0g)_{\lambda}.$$

To establish the relation with the scattering amplitude, consider for instance the first term on the right-hand side of (9). Assume a factorisation of the pair potentials

V_d into $V_d = X_d X'_d$ such that $M^\alpha(X'_d, \lambda)$ and $M^\beta(X_d, \lambda)$ exist. One may then write

$$\sum_{d\neq a} c_{\beta\alpha}(\lambda)\langle\lambda, \boldsymbol{\omega}_\beta; \beta|V_d|\lambda, \boldsymbol{\omega}_\alpha; \alpha\rangle = \sum_{d\neq a} c_{\beta\alpha}(\lambda)\langle\lambda, \boldsymbol{\omega}_\beta; \beta|E_\beta V_d E_\alpha|\lambda, \boldsymbol{\omega}_\alpha; \alpha\rangle$$
$$= \sum_{d\neq a} c_{\beta\alpha}(\lambda)\int \mathrm{d}^3x_d\, \mathrm{d}^3y_d \langle\lambda, \boldsymbol{\omega}_\beta; \beta|E_\beta X_d|\boldsymbol{x}_d, \boldsymbol{y}_d\rangle\langle\boldsymbol{x}_d, \boldsymbol{y}_d|X'_d E_\alpha|\lambda, \boldsymbol{\omega}_\alpha; \alpha\rangle. \tag{29}$$

The two factors in the integrand are (up to some multiplicative constants $d_\beta(\lambda)$ and $d_\alpha(\lambda)$) nothing but the integral kernels of the operators $M^\beta(X_d, \lambda)$ and $M^\alpha(X'_d, \lambda)^*$ respectively, if these operators are viewed as integral operators from $L^2(\mathbb{R}^6)$ to $L^2(S^2)$ (if $\beta \neq 0$) or $L^2(\mathscr{E}^5)$ (if $\beta = 0$) and from $L^2(S^2)$ to $L^2(\mathbb{R}^6)$ respectively. The integration in (29) corresponds to the composition of the two kernels, i.e. to multiplication of the two integral operators. Thus the expression (29) is proportional to the kernel of the integral operator

$$R^{(1)}_{\beta\alpha}(\lambda) = -2\pi\mathrm{i} \sum_{d\neq a} M^\beta(X_d, \lambda)M^\alpha(X'_d, \lambda)^*. \tag{30}$$

The constant of proportionality turns out to be equal to $(-2\pi\mathrm{i})^{-1}C_{\beta\alpha}(\lambda)d_\beta(\lambda)d_\alpha(\lambda) = -2\pi\mathrm{i}k_\alpha^{-1}$ (see remark 1).

In the same way, by virtue of (18) and the definition of Y_z, the second term on the right-hand side of (9) is given by $(-2\pi\mathrm{i}k_\alpha^{-1})$ times the integral kernel of

$$R^{(2)}_{\beta\alpha}(\lambda) = \lim_{\varepsilon\to+0}(-2\pi\mathrm{i})\sum_c{}' \sum_{d\neq a} M^\beta(A_c, \lambda)[J_{\lambda+\mathrm{i}\varepsilon}(\hat{I}+\hat{D}_{\lambda+\mathrm{i}\varepsilon})^{-1}K_{\lambda+\mathrm{i}\varepsilon}]^{cd}M^\alpha(V_dC_d^{-1}, \lambda)^*. \tag{31}$$

Here $M^\alpha(V_dC_d^{-1}, \lambda)^*$ maps $L^2(S^2)$ into $L^2(\mathbb{R}^6)$, the operator in the middle acts in $L^2(\mathbb{R}^6)$, and $M^\beta(A_c, \lambda)$ maps $L^2(\mathbb{R}^6)$ into $L^2(S^2)$ or $L^2(\mathscr{E}^5)$.

Let us now define

$$R_{\beta\alpha}(\lambda) = R^{(1)}_{\beta\alpha}(\lambda) + R^{(2)}_{\beta\alpha}(\lambda). \tag{32}$$

The preceding discussion shows that the scattering amplitude $f_{\beta\alpha}(\lambda)$ is equal to $(-2\pi\mathrm{i}k_\alpha^{-1})$ times the integral kernel of the operator $R_{\beta\alpha}(\lambda)$. Since the integral over both variables of the absolute square of the kernel of an integral operator is equal to the Hilbert–Schmidt norm of this operator, we find from (10) that

$$\bar{\sigma}_{\beta\alpha}(\lambda) = \frac{\pi}{2\mu_\alpha(\lambda-\lambda_\alpha)}\|R_{\beta\alpha}(\lambda)\|^2_{\mathrm{HS}}. \tag{33}$$

To prove the finiteness of $\bar{\sigma}_{\beta\alpha}(\lambda)$ and its continuity as a function of λ, it therefore suffices to show that $R^{(1)}_{\beta\alpha}(\lambda)$ and $R^{(2)}_{\beta\alpha}(\lambda)$ belong to the Hilbert–Schmidt class and depend continuously on λ in the Hilbert–Schmidt norm. This will be the content of the next section.

Remark 1. The relation (33) between the averaged total scattering cross section and the Hilbert–Schmidt norm of $R_{\beta\alpha}(\lambda)$ can be rigorously derived from the general principles of scattering theory by using a stationary expression for the scattering operator and lemma 4 (iv). The operator $R_{\beta\alpha}(\lambda)$ defined in (30)–(32) is related to the scattering matrix $S_{\beta\alpha}(\lambda)$ at energy λ by $R_{\beta\alpha}(\lambda) = S_{\beta\alpha}(\lambda) - I_\alpha$, where I_α denotes the identity operator in $L^2(S^2)$. The normalisation in deriving (33) is chosen such that the total cross section for a well collimated initial state in channel α having energy distribution $\varphi(\lambda)$ ($\alpha \div a$, $\lambda = \lambda_\alpha + \eta_a k_a^2$) is $\Sigma_\beta \int \varphi(\lambda)\bar{\sigma}_{\beta\alpha}(\lambda)\,d\lambda$.

4. Properties of the scattering cross section

In this section we shall use the results of § 3 to show that $R_{\beta\alpha}(\lambda) \in \mathscr{B}_2$ and is continuous in Hilbert–Schmidt norm, for each two-body incoming channel α and all final channels β, and for all energies λ except possibly those in $\Gamma_0 \cup \Theta$. For this we shall treat $R^{(1)}_{\beta\alpha}(\lambda)$ and $R^{(2)}_{\beta\alpha}(\lambda)$ separately. As before, we let $\alpha \div a$ and consider various possibilities of β, case by case.

Case I. $0 \neq \beta \div a$. In (30), we write $X_d = B_d$ and $X'_d = A_d$, to get $R^{(1)}_{\beta\alpha}(\lambda) = -2\pi i \,\Sigma_{d\neq a} M^\beta(B_d, \lambda)M^\alpha(A_d, \lambda)^*$. By lemma 4 (ii), $M^\beta(B_d, \lambda)$ and $M^\alpha(A_d, \lambda)$ belong to $\mathscr{B}_4$ and are continuous in $\mathscr{B}_4$ norm for all λ. It follows that $R^{(1)}_{\beta\alpha}(\lambda)$ is Hilbert–Schmidt and continuous in Hilbert–Schmidt norm (see Kato (1971) for more details on $\mathscr{B}_4$).

Case II. $0 \neq \beta \div b \neq a \div \alpha$. In this case we rewrite (30) as

$$R^1_{\beta\alpha}(\lambda) = -2\pi i \sum_{d\neq a} [M^\beta(C_b^3 W_{1d}^{1/2}, \lambda)M^\alpha(C_b^{-3}\rho_d^4 |W_{1d}|^{1/2}, \lambda)^* + M^\beta(C_b^2, \lambda)M^\alpha(C_b^{-2}V_{2d}, \lambda)^*] \tag{34}$$

where ρ_d means multiplication by $\rho(x_d) = (1+|x_d|)^{-1/2-\delta/4}$. By (27) and (26), $M^\beta(C_b^3 W_{1d}^{1/2}, \lambda) = M^\beta(C_b^3, \lambda)E_\beta W_{1d}^{1/2}$. Since $E_\beta W_{1d}^{1/2} \in \mathscr{B}(\mathscr{H})$ by lemma 2 and $\rho(y_b)^3 \leq \text{constant}\, \rho(x_b)^{-3}\rho(x_a)^3$ by (A10), we get from lemma 4 (i) ($m = 3$) that $M^\beta(C_b^3 W_{1d}^{1/2}, \lambda) \in \mathscr{B}_2$. Similarly $C_b^{-3}\rho_d^4 |W_{1d}|^{1/2} \leq \rho_a^{-3}\rho_d |W_{1d}|^{1/2}$, and since by the Hölder inequality $\rho(x_d)|W_{1d}(x_d)|^{1/2} \in L^2(\mathbb{R}^3)$, the second factor in the first term of (34) is also in $\mathscr{B}_2$. For the second term we notice that $M^\alpha(C_b^{-2}V_{2d}, \lambda) = M^\alpha(C_a^2, \lambda)E_a C_a^{-2}C_b^{-2}V_{2d}$ and that $E_a C_a^{-2}C_b^{-2}V_{2d} \in \mathscr{B}(\mathscr{H})$ by (S3), since $|\rho(y_a)^{-2}\rho(y_b)^{-2}V_{2d}(x_d)| \leq \text{constant}\, \rho(x_a)^{-4}$ by (A10). We then have $M^\beta(C_b^2, \lambda)M^\alpha(C_b^{-2}V_{2d}, \lambda)^* \in \mathscr{B}_2$ by lemma 4 (ii). The continuity in $\mathscr{B}_2$ norm of both terms in (34) follows from lemma 4.

Case III. $\beta = 0$. Using the spectral representation of H_0 given by U_d^0 and setting $A_{1d}(x_d) = |V_{1d}(x_d)|^{1/2}$, we have

$$R^1_{0\alpha}(\lambda) = -2\pi i \sum_{d\neq a} [M_d^0(B_{1d}, \lambda)M^\alpha(A_{1d}, \lambda) + U_{da}M_a^0(\rho_a^2, \lambda)M^\alpha(\rho_a^{-2}V_{2b}, \lambda)^*].$$

Hence $R^{(1)}_{0\alpha}(\lambda) \in \mathscr{B}_2$ by virtue of lemma 4 (iii) and lemma 4 (i) (with $m = 0$ and $m = 2$). The continuity of $R^{(1)}_{0\alpha}(\lambda)$ in Hilbert–Schmidt norm follows from the fact that both $M_d^0(B_{1d}, \lambda)$ and $M_a^0(\rho_a^2, \lambda)$ are strongly continuous in λ (lemma 4 (iii)) and the other factors are continuous in Hilbert–Schmidt norm (see lemma 8.23 of AJS).

Remark 2. From the above calculations it is clear that $V_c \in L^{3/2}_{\text{loc}}(\mathbb{R}^3)$ is the best we can do in order to ensure $R^{(1)}_{\beta\alpha}(\lambda) \in \mathscr{B}_2$. On the other hand, only for the elastic or simple inelastic scattering, i.e. in case I, do we need the fully decay property of V_c, namely $V_c(x) \sim |x|^{-2-\delta}$ at infinity. For rearrangement scattering (case II), if we assume that $W_{1c} = 0$, then

$$R^{(1)}_{\beta\alpha}(\lambda) = -2\pi i \sum_{d\neq a} M^\beta(\rho_a^3, \lambda)M^\alpha(\rho_a^{-3}V_{2d}, \lambda)^*$$

is in $\mathscr{B}_2$ even if $V_{2d}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-1/2-\varepsilon}$ at infinity, which can be seen by using the result of remark 7. On the other hand for break-up scattering (case III) it is necessary to have $V_{2d}(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-3/2-\varepsilon}$ at infinity.

For $R^{(2)}_{\beta\alpha}(\lambda)$ as given in (31), we shall consider the limits as $\varepsilon \to +0$ of those entries of the operator-valued matrices $M^\beta(A_c, \lambda)J_{\lambda+i\varepsilon}$ and $K_{\lambda+i\varepsilon}M^\alpha(V_dC_d^{-1}, \lambda)$ that actually occur in (31). The Hilbert–Schmidt nature of $R^{(2)}_{\beta\alpha}(\lambda)$ for λ in $\mathbb{R}\backslash(\Gamma_0 \cup \Theta)$ will follow if we can show that the first of these limits exists in some sense and the second one is Hilbert–Schmidt, since the limit of $(\hat{I} + \hat{D}_{\lambda+i\varepsilon})^{-1}$ as $\varepsilon \to +0$ exists in norm by proposition 1. The continuity in λ of $R^{(2)}_{\beta\alpha}(\lambda)$ will result if e.g. the first limit is strongly continuous and the second one continuous in Hilbert–Schmidt norm, by using lemma 8.23 of AJS and the norm continuity of $(\hat{I} + \hat{D}_{\lambda+i0})^{-1}$ on $\mathbb{R}\backslash(\Gamma_0 \cup \Theta)$.

We first treat the relevant entries of $M^\beta(A_c, \lambda)J_{\lambda+i\varepsilon}$. By (12) we have

$$M^\beta(A_c, \lambda)[J_{\lambda+i\varepsilon}(f_0 \oplus f_1)]_c = M^\beta(A_c, \lambda)f_{0,c} + M^\beta(A_c, \lambda)B_cE_cR^c_{\lambda+i\varepsilon}C_cf_{1,c}.$$

The first term on the right-hand side is independent of ε, and by lemma 4 $M^\beta(A_c, \lambda)$ is a bounded operator which is at least strongly continuous in λ for all β. For the second term on the right-hand side we first assume $0 \neq \beta \div b$ and notice that $c \neq b$ (as expressed by the symbol Σ'_c in (31). Using (26), we write

$$M^\beta(A_c, \lambda)B_cE_cR^c_{\lambda+i\varepsilon}C_c = M^\beta(\rho_c^2A_cC_c^{-1}, \lambda)[W_c^{1/2}E_c]C_cE_cR^c_{\lambda+i\varepsilon}C_c.$$

The first factor is Hilbert–Schmidt by lemma 4 (i), the second one is bounded by lemma 2 and the last one has a norm limit as $\varepsilon \to +0$ for all $\lambda \in \mathbb{R}\backslash\Theta_c$ by lemma 6 (ii). The continuity in λ is obtained similarly.

For $\beta = 0$ we let F^c_μ be the spectral projection of $\eta_c\boldsymbol{K}_c^2$ for the interval $[0, \mu]$ and observe that $E^0_\Delta = E^0_\Delta F^c_\mu$ for every interval $\Delta \subseteq [0, \mu]$. Using this, (28) and (26) one has for $\lambda \in \Delta$: $M^0_c(A_c, \lambda) = M^0_c(E^0_\Delta F^c_\mu A_c, \lambda) = M^0_c(A_c, \lambda)F^c_\mu$. Thus

$$\begin{aligned} &M^0_c(A_c, \lambda)B_cE_cR^c_{\lambda+i\varepsilon}C_c \\ &\quad = \sum_{\gamma \div c} M^0_c(A_c, \lambda)[B_cE_c][F^c_\mu(\eta_c\boldsymbol{K}_c^2 + \lambda_\gamma - \lambda - i\varepsilon)^{-1}E_\gamma C_c]. \end{aligned} \tag{35}$$

Since $A_c(\cdot) \in L^2(\mathbb{R}^3)$, $M^0_c(A_c, \lambda)$ is bounded by lemma 4 (iii). Furthermore $B_cE_c \in \mathscr{B}(\mathscr{H})$ by lemma 2. An appropriate choice of μ and Δ, as on p 666 of AJS, shows that the last factor in (35) has a norm-continuous boundary value for $\lambda \in \Delta$, leading to the strong continuity of the right-hand side of (35) for $\lambda \geqslant 0$.

Now we treat the term $K^{cd}_{\lambda+i\varepsilon}M^\alpha(V_dC_d^{-1}, \lambda)^*$, with $\alpha \div a \neq d$. From (13) and (26) we obtain

$$\begin{aligned} &K^{0,cd}_{\lambda+i\varepsilon}M^\alpha(V_dC_d^{-1}, \lambda)^* \\ &\quad = B_c(I - E_c)R^c_{\lambda+i\varepsilon}M^\alpha(V_d, \lambda)^* \\ &\quad = B_c(I - E_c)R^c_{\lambda+i\varepsilon}A_{1d}M^\alpha(B_{1d}, \lambda)^* + B_c(I - E_c)R^c_{\lambda+i\varepsilon}\rho_a^2M^\alpha(V_{2d}\rho_a^{-2}, \lambda)^*. \end{aligned} \tag{36}$$

That the first factors of the two summands in (36) have norm-continuous boundary values as $\varepsilon \to +0$ follows from lemma 7 (when $d = c$) and from the first part of the proof of lemma 3 (when $d \neq c$). On the other hand $B_{1d}(\cdot)$ and $V_{2d}(\cdot)$ are in $L^2(\mathbb{R}^3)$, and thus both $M^\alpha(B_{1d}, \lambda)^*$ and $M^\alpha(V_{2d}\rho_a^{-2}, \lambda)^*$ are Hilbert–Schmidt by lemma 4 (i).

Finally we have

$$K^{1,cd}_{\lambda+i\varepsilon}M^\alpha(V_dC_d^{-1},\lambda)^* = C_c^{-1}E_cC_dM^\alpha(V_dC_d^{-1},\lambda)^*$$
$$= E_cM^\alpha(V_dC_c^{-1},\lambda)^*$$
$$= [E_cW_d^{1/2}]M^\alpha(\rho_d^2A_dC_c^{-1},\lambda)^*.$$

By lemma 2, $E_cW_d^{1/2} \in \mathscr{B}(\mathscr{H})$. Since $\rho(x_d)^2A_d(x_d)\rho(y_c)^{-1} \leq \text{constant}\ \rho(x_d)A_d(x_d)\rho(x_a)^{-1}$ by (A10) and since $\rho(x_d)A_d(x_d) \in L^2(\mathbb{R}^3)$, and $\alpha \div a \neq d$, we have by lemma 4 (i) that $M^\alpha(\rho_d^2A_dC_c^{-1},\lambda)^* \in \mathscr{B}_2$. It is clear that $K^{cd}_{\lambda+i0}M^\alpha(V_dC_d^{-1},\lambda)^*$ is a continuous function of λ in Hilbert–Schmidt norm on $\mathbb{R}\backslash\Theta$.

Thus we have proven the following proposition.

Proposition 2. Let each pair potential V_c be as in (1), and assume (S1), (S2) and (S3). Let α be a two-body channel. Then $R_{\beta\alpha}(\lambda)$ is a Hilbert–Schmidt operator for each $\lambda \in (\lambda_\alpha, \infty)\backslash(\Gamma_0 \cup \Theta)$ and each open final channel, and $\{R_{\beta\alpha}(\lambda)\}$ is continuous in Hilbert–Schmidt norm on $(\lambda_\alpha, \infty)\backslash(\Gamma_0 \cup \Theta)$. The averaged total scattering cross section $\bar{\sigma}_{\alpha,\text{tot}}(\lambda)$ for the initial channel α is finite and is a continuous function of λ on $(\lambda_\alpha, \infty)\backslash(\Gamma_0 \cup \Theta)$.

Remark 3. The threshold set Θ appears in proposition 2 only because of the application of lemma 6 (ii) in the proof of lemma 3, which is concerned with the continuous bounded invertibility of $\hat{I} + \hat{D}_{\lambda+i0}$. This happens because, in order that the term $K^{1,cd}_zM^\alpha(V_dC_d^{-1},\lambda)^* = [E_cW_d^{1/2}]M^\alpha(\rho_d^2A_dC_c^{-1},\lambda)^*$ is Hilbert–Schmidt, we had to choose $C(y) = \rho(y) \equiv (1+|y|)^{-1/2-\delta/4}$. By taking $C(y) = (1+|y|)^{-1-\delta/2}$, one could avoid the threshold problem, but then W_{2c} would have to be zero in order that $R^{(2)}_{\beta\alpha}(\lambda) \in \mathscr{B}_2$.

Remark 4. We comment here briefly on some spectral properties of two- and three-body Hamiltonians.

(i) The hypotheses (S1) and (S2), for a potential V_a satisfying (1), are known to hold, except possibly for a discrete set of values of the coupling constant, if W_1 has compact support. In fact, by proposition 3.5 of Ginibre and Moulin (1974), $\omega^a_{\lambda+i0}$ has the eigenvalue -1 for at most a finite set of points λ, all of which are eigenvalues of finite multiplicity of $h_a = \gamma_a P_a^2 + V_a$ (except possibly $\lambda = 0$). If W_1 is of compact support, then h_a can have no positive eigenvalues (this follows by combining the results of Kato (1959) and of Amrein *et al* (1981)). The invertibility of $\omega^a_{\lambda+i0} + 1$ for $\lambda = 0$, except possibly for a discrete set of values of the coupling constant, is discussed in remark 10.18(*b*) of AJS. The decay assumption (S3) of the eigenfunctions of h_a (for negative eigenvalues) is satisfied by theorem XIII.40 of Reed and Simon (1978). By using the exponential decay of these eigenfunctions $e_\alpha(x)$, one can show by an iteration procedure as in the proof of lemma 1 that $(1+|x|)^\theta e_\alpha(x) \in D(|P|)$ for each $\theta \in \mathbb{R}$.

(ii) As regards the exceptional set Γ_0 for the three-body problem, one expects that all points in Γ_0 are eigenvalues of the three-body Hamiltonian H. This can be shown for negative λ in Γ_0 essentially as in proposition 7.2 of Ginibre and Moulin (1974). In particular this result implies absence of a singularly continuous spectrum of H in $(-\infty, 0)$. For the positive exceptional points the problem is not solved for our class of potentials; absence of a singular continuous spectrum has been shown under stronger assumptions on the pair potential by Faddeev (1965), Jafaev (1978), Sigal (1978) and Mourre (1981).

Appendix

In this appendix we present the more technical aspects of our proofs. We first give some simple properties of the pair potentials.

In (1) we assumed that $W_1 \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $p > \frac{3}{2} > q$. It is easy to see that the same class of potentials V is obtained if one assumes only $W_1 \in L^p(\mathbb{R}^3)$ for some $p > \frac{3}{2}$. In many parts of the proofs below it even suffices to know that $W_1 \in L^{3/2}(\mathbb{R}^3)$.

Let $|W|^{1/2}$ be the multiplication operator in $L^2(\mathbb{R}^3)$ by $|W_1(\boldsymbol{x}) + W_2(\boldsymbol{x})|$ and $P = -\mathrm{i}\boldsymbol{\nabla}_x$. Using the inequality $|a+b|^{1/2} \leq |a|^{1/2} + |b|^{1/2}$ and lemma 7.24 of AJS, one sees that, if $p > \frac{3}{2}$:

$$\||W|^{1/2}(|\boldsymbol{P}|+1)^{-1}\| \leq \||W_1|^{1/2}(|\boldsymbol{P}|+1)^{-1}\| + \||W_2|^{1/2}(|\boldsymbol{P}|+1)^{-1}\|$$
$$\leq \|W_1\|_p^{1/2}\|(|\boldsymbol{k}|+1)^{-1}\|_{2p} + \|W_2\|_\infty^{1/2} < \infty. \tag{A1}$$

In particular, $|W|^{1/2}(|\boldsymbol{P}|+1)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^3))$. If W_1 is only in $L^{3/2}(\mathbb{R}^3)$, one obtains by using the identity $\|A^*A\| = \|A\|^2$ and the Sobolev inequality (Kato 1971, lemma 1) that

$$\||W_1|^{1/2}(|\boldsymbol{P}|+1)^{-1}\| \leq \||W_1|^{1/2}(\boldsymbol{P}^2+1)^{-1/2}\|\|(\boldsymbol{P}^2+1)^{1/2}(|\boldsymbol{P}|+1)^{-1}\|$$
$$\leq \||W_1|^{1/2}(\boldsymbol{P}^2+1)^{-1}|W_1|^{1/2}\|^{1/2}$$
$$\leq \text{constant}\,\|W_1\|_{3/2}^{1/2}. \tag{A2}$$

Proof of lemma 1. We omit the subscripts α and a and denote by $\boldsymbol{Q}$ the position operator in $L^2(\mathbb{R}^3)$. By hypothesis, we have $e \in D((I+|\boldsymbol{Q}|)^{2+\delta}$, $e \in D(H) \subseteq D(|H|^{1/2}) = D(|\boldsymbol{P}|)$ and

$$\sum_{j=1}^{3} (P_j f, P_j e) + (|V|^{1/2} f, V^{1/2} e) = \lambda(f, e) \qquad \forall f \in D(|\boldsymbol{P}|). \tag{A3}$$

We notice that $D(|\boldsymbol{P}|) = \bigcap_{j=1}^{3} D(P_j)$ and that

$$\sum_{j=1}^{3} (P_j f, P_j g) = (|\boldsymbol{P}|f, |\boldsymbol{P}|g) \qquad \forall f, g \in D(|\boldsymbol{P}|). \tag{A4}$$

(i) Let $\psi: \mathbb{R}^3 \to [0, \infty)$ be a non-negative C_0^∞ function such that $\psi(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \geq 1$ and $\int \psi(\boldsymbol{x})\,\mathrm{d}^3x = 1$, and set $\psi_\varepsilon(\boldsymbol{x}) = \varepsilon^{-3}\psi(\varepsilon^{-1}\boldsymbol{x})$ for $\varepsilon \in (0,1)$. For $f \in L^2(\mathbb{R}^3)$, set $f_\varepsilon = \psi_\varepsilon * f$, i.e.

$$f_\varepsilon(\boldsymbol{x}) = \int \psi_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}^3y = \int \psi_\varepsilon(\boldsymbol{z}) f(\boldsymbol{x}-\boldsymbol{z})\,\mathrm{d}^3z. \tag{A5}$$

Then (see Adams 1975, lemma 2.18) $f_\varepsilon \in C^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $\|f_\varepsilon\| \leq \|f\|$ and $\|f - f_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

We apply this to $f = e$. We claim that, for each $\kappa \in [0, 2+\delta]$

$$e_\varepsilon \in D(P_j) \qquad \|P_j e - P_j e_\varepsilon\| \to 0 \qquad \text{as } \varepsilon \to 0 \tag{A6}$$

and

$$e_\varepsilon \in D(|\boldsymbol{Q}|^\kappa) \qquad \|(I+|\boldsymbol{Q}|)^\kappa (e - e_\varepsilon)\| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{A7}$$

(A6) is immediate if we notice that $P_j e_\varepsilon = \psi_\varepsilon * (P_j e)$ (the derivative $-\mathrm{i}\partial/\partial x_j$ may be interchanged with the integral in (A5)). To prove (A7), we observe that

$$|\boldsymbol{x}-\boldsymbol{y}| \leq \varepsilon < 1 \Rightarrow |(1+|\boldsymbol{x}|)^\kappa - (1+|\boldsymbol{y}|)^\kappa| \leq \varepsilon\kappa(2+|\boldsymbol{y}|)^\kappa. \tag{A8}$$

This implies that

$$|(1+|\boldsymbol{x}|)^\kappa e_\varepsilon(\boldsymbol{x})| \leqslant |[\psi_\varepsilon * (I+|\boldsymbol{Q}|)^\kappa e](\boldsymbol{x})| + \varepsilon\kappa \int \psi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})(2+|\boldsymbol{y}|)^\kappa |e(\boldsymbol{y})|.$$

Since $e \in D((I+|\boldsymbol{Q}|)^\kappa)$, we obtain that $(I+|\boldsymbol{Q}|)^\kappa e_\varepsilon \in L^2(\mathbb{R}^3)$. Similarly one finds that $|(1+|\boldsymbol{x}|)^\kappa[e(\boldsymbol{x})-e_\varepsilon(\boldsymbol{x})]|$

$$\leqslant \varepsilon\kappa \int \psi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})(2+|\boldsymbol{y}|)^\kappa|e(\boldsymbol{y})| + |(1+|\boldsymbol{x}|)^\kappa e(\boldsymbol{x}) - [\psi_\varepsilon * (I+|\boldsymbol{Q}|)^\kappa e](\boldsymbol{x})|.$$

Since the right-hand side converges strongly to zero as $\varepsilon \to 0$, we get the second assertion in (A7).

We remark the following consequence of (A6):

$$\chi(\cdot)e_\varepsilon(\cdot) \in D(P_j) \qquad \text{for each } \chi \in C_0^\infty(\mathbb{R}^3). \tag{A9}$$

(ii) Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $0 \leqslant \varphi(\boldsymbol{x}) \leqslant 1$, $\varphi(\boldsymbol{x})=1$ for $|\boldsymbol{x}|\leqslant 1$ and $\varphi(\boldsymbol{x})=0$ for $|\boldsymbol{x}|\geqslant 2$, and let Φ_m be the multiplication operator by $\varphi_m(\boldsymbol{x}) := \varphi(m^{-1}\boldsymbol{x})$ $(m \geqslant 1)$. Then, since $(I+|\boldsymbol{Q}|)^\nu\Phi_m$ is a bounded operator and $(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2 e_\varepsilon \in D(P_j)$ by (A9)

$$\begin{aligned}
\sum_{j=1}^3 &\|(I+|\boldsymbol{Q}|)^\nu\Phi_m P_j e\|^2 \\
&= \lim_{\varepsilon\to 0} \sum_{j=1} ((I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2 P_j e_\varepsilon, P_j e) \\
&= \lim_{\varepsilon\to 0} \sum_{j=1}^3 \{(P_j(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2 e_\varepsilon, P_j e) + ([(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2, P_j]e_\varepsilon, P_j e)\} \\
&= \lim_{\varepsilon\to 0} \{\lambda((I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2 e_\varepsilon, e) - (|V|^{1/2}(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2 e_\varepsilon, V^{1/2}e)\} \\
&\quad + \lim_{\varepsilon\to 0}\sum_{j=1}^3 ([(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2, P_j]e_\varepsilon, P_j e) \equiv \lim_{\varepsilon\to 0}(t_\varepsilon^{(1)} + t_\varepsilon^{(2)} + t_\varepsilon^{(3)}).
\end{aligned}$$

We estimate each term on the right-hand side. For $t_\varepsilon^{(1)}$ and $t_\varepsilon^{(2)}$ we obtain

$$\begin{aligned}
|t_\varepsilon^{(1)}| &\leqslant |\lambda|\,\|\varphi\|_\infty^2 \|(I+|\boldsymbol{Q}|)^\nu e_\varepsilon\|\,\|(I+|\boldsymbol{Q}|)^\nu e\| \\
|t_\varepsilon^{(2)}| &\leqslant \|\varphi\|_\infty^2 \|W^{1/2}(|\boldsymbol{P}|+1)^{-1}\|^2 \|(|\boldsymbol{P}|+1)e_\varepsilon\|\,\|(|\boldsymbol{P}|+1)(I+|\boldsymbol{Q}|)^{2\nu-2-\delta}e\|.
\end{aligned}$$

For $t_\varepsilon^{(3)}$ we notice that

$$[(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m^2, P_j] = -\mathrm{i}\frac{Q_j}{|\boldsymbol{Q}|}(I+|\boldsymbol{Q}|)^{2\nu}\Phi_m\left((I+|\boldsymbol{Q}|)^{-1}\Phi_m + \frac{2}{m}\Phi_m^{(j)}\right)$$

where $\Phi_m^{(j)}$ is multiplication by $(\partial\varphi/\partial x_j)(m^{-1}\boldsymbol{x})$. Hence

$$|t_\varepsilon^{(3)}| \leqslant \sum_{j=1}^3 \|\varphi\|_\infty\left(\|\varphi\|_\infty + \frac{2}{m}\|\mathrm{grad}\ \varphi\|_\infty\right)\|(I+|\boldsymbol{Q}|)^{2\nu-\theta}e_\varepsilon\|\,\|(I+|\boldsymbol{Q}|)^\theta P_j e\|.$$

We first take $\theta=0$ and $\nu \leqslant 1+\frac{1}{2}\delta$ (i.e. $2\nu-2-\delta\leqslant 0$, $2\nu-\theta\leqslant 2+\delta$). In view of (A1), (A6) and (A7), the above estimates then imply that

$$\sum_{j=1}^3 \|(I+|\boldsymbol{Q}|)^\nu\Phi_m P_j e\|^2 \leqslant c < \infty \qquad \forall m \geqslant 1$$

where c is independent of m. Letting $m \to \infty$, this implies (for instance by the monotone convergence theorem) that $(I+|\boldsymbol{Q}|)^{\nu}P_j e \in L^2(\mathbb{R}^3)$ for each j. Since $P_j(I+|\boldsymbol{Q}|)^{\nu}e = (I+|\boldsymbol{Q}|)^{\nu}P_j e - \mathrm{i}\nu Q_j|\boldsymbol{Q}|^{-1}(I+|\boldsymbol{Q}|)^{\nu-1}e$, we have $(I+|\boldsymbol{Q}|)^{\nu}e \in \bigcap_j D(P_j) \equiv D(|\boldsymbol{P}|)$ for all $\nu \leqslant 1+\frac{1}{2}\delta$.

We next take $\theta = 1+\frac{1}{2}\delta$ and $\nu \leqslant \frac{3}{4}(1+\frac{1}{2}\delta)$ (i.e. $2\nu-2-\delta \leqslant 1+\frac{1}{2}\delta$ and $2\nu-\theta \leqslant 2+\delta$) and find similarly that $(I+|\boldsymbol{Q}|)^{\nu}P_j e \in L^2(\mathbb{R}^3)$ and $(I+|\boldsymbol{Q}|)^{\nu}e \in D(|\boldsymbol{P}|)$. The assertion of the lemma is obtained by iterating this procedure, taking in the nth step $\theta = (1-2^{-n})(2+\delta)$ and $\nu \leqslant (1-2^{-n-1})(2+\delta)$.

Proof of lemma 2. (i) We identify $L^2(\mathbb{R}^6)$ with the spectral representation of the multiplication operator by $\boldsymbol{y}_b$, i.e. $L^2(\mathbb{R}^6) = L^2(\mathbb{R}^3, L^2_{x_b}(\mathbb{R}^3); \mathrm{d}^3 y_b)$, the space of all square-integrable $L^2_{x_b}(\mathbb{R}^3)$-valued functions of $\boldsymbol{y}_b \in \mathbb{R}^3$. In this representation of $L^2(\mathbb{R}^6)$, the operator $|W_a|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}$ is decomposable, i.e. given by a family $\{F(\boldsymbol{y}_b)\}$ of operators acting in $L^2_{x_b}(\mathbb{R}^3)$. $F(\boldsymbol{y}_b)$ is as follows

$$\begin{aligned} F(\boldsymbol{y}_b) &= |W_a(\mu \boldsymbol{Q}_b + \nu \boldsymbol{y}_b)|^{1/2}(|\boldsymbol{P}_b|+1)^{-1} \\ &= \exp(\mathrm{i}\nu \boldsymbol{y}_b \cdot \boldsymbol{P}_b)|W_a(\mu \boldsymbol{Q}_b)|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}\exp(-\mathrm{i}\nu \boldsymbol{y}_b \cdot \boldsymbol{P}_b) \end{aligned}$$

where $\mu \neq 0$. We have

$$\||W_a|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}\| = \sup_{\boldsymbol{y}_b \in \mathbb{R}^3} \|F(\boldsymbol{y}_b)\| = \||W_a(\mu \boldsymbol{Q}_b)|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}\|$$

which is finite by (A1) or (A2) since $\mu \neq 0$. Thus $|W_a|^{1/2}(|\boldsymbol{P}_b|+1)^{-1} \in \mathcal{B}(\mathcal{H})$.

The boundedness of the other operators in (i) now follows easily. For example $\|B_a(|\boldsymbol{P}_b|+1)^{-1}\| \leqslant \||W_a|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}\| < \infty$. Also, setting $h_b = \gamma_b \boldsymbol{P}_b^2 + V_b$,

$$\||W_a|^{1/2}E_b\| \leqslant \||W_a|^{1/2}(|\boldsymbol{P}_b|+1)^{-1}\|\|(|\boldsymbol{P}_b|+1)(|h_b|+1)^{-1/2}\|\|(|h_b|+1)^{1/2}E_b\|$$

which is finite since $\|(|h_b|+1)^{1/2}E_b\| = \sup_{\beta \div b}(|\lambda_\beta|+1)^{1/2} < \infty$.

(ii) We shall use the following inequality: if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, $\mu, \nu \in \mathbb{R}$, then

$$(1+|\mu \boldsymbol{u}+\nu \boldsymbol{v}|) \leqslant \max\{1,|\mu|,|\nu|\}(1+|\boldsymbol{u}|+|\boldsymbol{v}|) \leqslant \max\{1,|\mu|,|\nu|\}(1+|\boldsymbol{u}|)(1+|\boldsymbol{v}|). \tag{A10}$$

If $c \neq d$, we may write $\boldsymbol{y}_a = \mu \boldsymbol{x}_c + \nu \boldsymbol{x}_d$, so that

$$(1+|\boldsymbol{y}_a|)^{1/2+\delta/4} \leqslant \text{constant}\,(1+|\boldsymbol{x}_c|)^{1/2+\delta/4}(1+|\boldsymbol{x}_d|)^{1/2+\delta/4}.$$

In view of the definition of A_c, this leads to

$$\begin{aligned} \|C_a^{-1}C_b^{-1}A_c E_d f\| &\leqslant \text{constant}\,\||W_c|^{1/2}(1+|\boldsymbol{Q}_d|)^{1+\delta/2}E_d f\| \\ &\leqslant \text{constant}\,\||W_c|^{1/2}(|\boldsymbol{P}_d|+1)^{-1}\|\|(|\boldsymbol{P}_d|+1)(1+|\boldsymbol{Q}_d|)^{1+\delta/2}E_d\|\|f\|. \end{aligned}$$

Now

$$\begin{aligned} \|(|\boldsymbol{P}_d|+1)(1+|\boldsymbol{Q}_d|)^{1+\delta/2}E_d\| &\leqslant \sum_{\alpha \div d} \|(|\boldsymbol{P}_d|+1)(1+|\boldsymbol{Q}_d|)^{1+\delta/2}E_\alpha\| \\ &= \sum_{\alpha \div d} \|(|\boldsymbol{P}_d|+1)(1+|\boldsymbol{Q}_d|)^{1+\delta/2}e_\alpha\|. \end{aligned} \tag{A11}$$

Therefore, by the result of (i) and lemma 1, $C_a^{-1}C_b^{-1}A_c E_d \in \mathcal{B}(\mathcal{H})$. The boundedness of the remaining two operators in (ii) follows from this, the fact that $C_c E_c = E_c C_c$ and the self-adjointness of all operators involved.

(iii) By writing $\boldsymbol{y}_a = \kappa \boldsymbol{y}_b + \tau \boldsymbol{x}_a$, we get from (A10) as above that

$$\|C_a^{-1}C_b E_a\| \leqslant \text{constant}\|(1+|\boldsymbol{Q}_a|)^{1/2+\delta/4}E_a\| \leqslant \text{constant} \sum_{\alpha \div a} \|(1+|\boldsymbol{Q}_a|)^{1/2+\delta/4}e_\alpha\| < \infty.$$

Remark 5. It is seen from part (i) of the preceding proof that $|W_a|^{1/2}E_b$ may be finite even if the two-body Hamiltonian h_b has an infinite number of bound states. It suffices to know that all eigenvalues of h_b are lying in a compact set.

The next three lemmas contain the preliminary results needed to prove lemmas 3 and 4. We set $W_z^{cd} = B_c(H_0 - z)^{-1}A_d$ and let $W_z = \{W_z^{cd}\}$ be the associated operator in $\mathscr{H}$.

Lemma 5. Let $\operatorname{Im} z \neq 0$. Then
(i) W_z, Y_z, G_z, K_z^0, K_z^1 are all in $\mathscr{B}(\mathscr{H})$,
(ii) $D_{z,ik} \in \mathscr{B}(\mathscr{H})$ for all i, $k = 0, 1$.

Proof. (i) We have $B_c(H_0+1)^{-1/2} = [B_c(|\boldsymbol{P}_c|+1)^{-1}][(|\boldsymbol{P}_c|+1)(H_0+1)^{-1/2}]$. The first factor is in $\mathscr{B}(\mathscr{H})$ by lemma 2 (i), and the second one by the definition (2) of H_0. Similarly one obtains $(H_0+1)^{-1/2}A_d \in \mathscr{B}(\mathscr{H})$. Hence, if $\operatorname{Im} z \neq 0$,

$$B_c(H_0-z)^{-1}A_d = [B_c(H_0+1)^{-1/2}][(H_0+1)(H_0-z)^{-1}][(H_0+1)^{-1/2}A_d] \in \mathscr{B}(\mathscr{H}).$$

From (A2) we also obtain that there is a constant $\kappa(z)$ such that

$$\|B_c(H_0-z)^{-1}A_d\| \leq \kappa(z)\|B_c(\cdot)\|_3\|A_d(\cdot)\|_3. \tag{A12}$$

Similarly we have

$$Y_z^{cd} = B_c(H-z)^{-1}C_d = [B_c(H_0+1)^{-1/2}][(H_0+1)^{1/2}(H-z)^{-1}]C_d \in \mathscr{B}(\mathscr{H})$$

since $C_d \in \mathscr{B}(\mathscr{H})$ and H_0 is form bounded relative to H. An identical line of reasoning works for G_z and K_z^0 since $(H_0+1)^{1/2}(H_c-z)^{-1} \in \mathscr{B}(\mathscr{H})$ for all c. That $K_z^1 \in \mathscr{B}(\mathscr{H})$ is contained in lemma 2 (iii).

(ii) It suffices to show that $D_{z,00} \in \mathscr{B}(\mathscr{H})$, for the boundedness of $D_{z,10}$ is a consequence of lemma 2 while that of G_z has been shown in part (i). For this we observe that

$$\begin{aligned} B_c(I-E_c)R_z^cA_d &= [B_c(H_0+1)^{-1/2}][(H_0+1)^{1/2}R_z^c(H_0+1)^{1/2}][(H_0+1)^{-1/2}A_d] \\ &\quad - B_cE_c[R_z^c(H_0+1)^{1/2}][(H_0+1)^{-1/2}A_d] \end{aligned}$$

which is in $\mathscr{B}(\mathscr{H})$ by (i), the form boundedness of H_0 relative to H_c and lemma 2 (i).

Lemma 6. (i) For all c, d, $W_{\lambda+i\varepsilon}^{cd}$ converges in operator norm to $W_{\lambda+i0}^{cd}$ as $\varepsilon \to +0$, uniformly in $\mathbb{R}$, and for $c \neq d$, $W_z^{cd} \in \mathscr{B}_\infty(\mathscr{H})$ for all z.

(ii) $C_dR_z^dE_dC_d \in \mathscr{B}_\infty(\mathscr{H})$ and has norm boundary values as $z \to \lambda + i0$ for all $\lambda \in \mathbb{R}\backslash\Theta_d$. Moreover, the convergence is uniform in every compact subset of $\mathbb{R}\backslash\Theta_d$.

Proof. (i) Denoting $U_t = \exp(-iH_0t)$ and setting for simplicity $H_0 = \boldsymbol{P}_c^2 + \boldsymbol{K}_c^2$, we have by using the commutativity of $\boldsymbol{Q}_c$ and $\boldsymbol{K}_c$ and equation (13.4) of AJS

$$U_t^*B_cU_tA_d = \exp(i\boldsymbol{P}_c^2t)B_c\exp(-i\boldsymbol{P}_c^2t)A_d = \exp(i\boldsymbol{Q}_c^2/4t)B_c(2\boldsymbol{P}_ct)\exp(-i\boldsymbol{Q}_c^2/4t)A_d. \tag{A13}$$

The last expression defines a decomposable operator in $L^2(\mathbb{R}^6) \equiv L^2(\mathbb{R}^3, L_{x_c}^2(\mathbb{R}^3); d^3y_c)$. Thus $U_t^*B_cU_tA_d = \{F(y_c)\}$, with

$$\begin{aligned} F(y_c) &= \exp(i\boldsymbol{Q}_c^2/4t)B_c(2\boldsymbol{P}_ct)A_d(\mu\boldsymbol{Q}_c + \nu y_c)\exp(-i\boldsymbol{Q}_c^2/4t) \\ &= \exp(i\boldsymbol{Q}_c^2/4t)\exp[i(\nu/\mu)\boldsymbol{P}_c\cdot y_c]B_c(2\boldsymbol{P}_ct)A_d(\mu\boldsymbol{Q}_c) \\ &\quad \times\exp[-i(\nu/\mu)\boldsymbol{P}_c\cdot y_c]\exp(-i\boldsymbol{Q}_c^2/4t). \end{aligned}$$

We notice that μ is never 0 in the change of variables relation $x_d = \mu x_c + \nu y_c$ and get, again using equation (13.4) of AJS, that

$$\begin{aligned}\|B_c U_t A_d\| &= \|B_c(2\boldsymbol{P}_c t)A_d(\mu \boldsymbol{Q}_c)\| \\ &= \|\exp(-\mathrm{i}\boldsymbol{Q}_c^2/4t)\exp(\mathrm{i}\boldsymbol{P}_c^2 t)B_c \exp(-\mathrm{i}P_c^2 t)\exp(\mathrm{i}\boldsymbol{Q}_c^2/4t)A_d(\mu\boldsymbol{Q}_c)\| \\ &= \|B_c \exp(-\mathrm{i}\boldsymbol{P}_c^2 t)A_d(\mu\boldsymbol{Q}_c)\|. \end{aligned} \tag{A14}$$

Since $B_c(H_0 - z)^{-1}A_d = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}B_c U_t A_d \, \mathrm{d}t$ for $\operatorname{Im} z > 0$, it follows that we shall have the first part of (i) if we can show that $\|B_c U_t A_d\| \in L^1(\mathbb{R})$. This, by (A14), reduces to showing that the two-body object $\|B_c(\boldsymbol{Q})\exp(-\mathrm{i}\boldsymbol{P}^2 t)A_d(\mu\boldsymbol{Q})\|$ is in $L^1(\mathbb{R})$. This follows from a result in § 6 of Kato (1966), since by our hypothesis on V_c, one has that $|V_c(\cdot)|^{1/2} \in L^{2p}(\mathbb{R}^3) \cap L^{2q}(\mathbb{R}^3)$ with $1 \leq q < \frac{3}{2} < p$.

Let $c \neq d$, $\operatorname{Im} z \neq 0$ and $W_{z,n}^{cd} := \varphi_n(\boldsymbol{Q}_c)(H_0 - z)^{-1}\psi_n(\boldsymbol{Q}_d)$, where φ_n and ψ_n are $\mathcal{S}(\mathbb{R}^3)$ functions such that $\|B_c(\cdot) - \varphi_n(\cdot)\|_3 \to 0$ and $\|A_d(\cdot) - \psi_n(\cdot)\|_3 \to 0$ as $n \to \infty$. Since φ_n, $\psi_n \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for each n, it follows from lemma 16.3(a) of AJS that $W_{z,n}^{cd} \in \mathcal{B}_\infty(\mathcal{H})$. On the other hand,

$$\begin{aligned}\|W_z^{cd} - W_{z,n}^{cd}\| &\leq \|(B_c - \varphi_n(\boldsymbol{Q}_c))(H_0 - z)^{-1}A_d\| + \|(\varphi_n(\boldsymbol{Q}_c)(H_0 - z)^{-1}(A_d - \psi_n(\boldsymbol{Q}_d))\| \\ &\leq \text{constant}[\|B_c(\cdot) - \varphi_n(\cdot)\|_3\|A_d(\cdot)\|_3 + \|\varphi_n(\cdot)\|_3\|A_d(\cdot) - \psi_n(\cdot)\|_3]\end{aligned}$$

by (A12) and thus converges to zero as $n \to \infty$, showing that $W_z^{cd} \in \mathcal{B}_\infty(\mathcal{H})$ if $\operatorname{Im} z \neq 0$. Since $W_{\lambda+\mathrm{i}\varepsilon}^{cd}$ converges to $W_{\lambda+\mathrm{i}0}^{cd}$ in norm, we also have $W_{\lambda+\mathrm{i}0}^{cd} \in \mathcal{B}_\infty(\mathcal{H})$.

(ii) By (7) and since $\eta_d \neq 0$,

$$\begin{aligned}C_d R_z^d E_d C_d &= \sum_{\alpha \div d} C_d(\eta_d \boldsymbol{K}_d^2 + \lambda_\alpha - z)^{-1}E_\alpha C_d \\ &= \eta_d^{-1} \sum_{\alpha \div d} E_\alpha C_d(\boldsymbol{K}_d^2 - \eta_d^{-1}(z - \lambda_\alpha))^{-1}C_d.\end{aligned} \tag{A15}$$

By proposition 10.23 of AJS, the operator $\rho(\boldsymbol{Q})(\boldsymbol{K}^2 - z')^{-1}\rho(\boldsymbol{Q})$ in $L^2(\mathbb{R}^3)$ ($\boldsymbol{K} = -\mathrm{i}$ grad) is compact and has boundary values in operator norm as $z' \to \mu + \mathrm{i}0$ with $\mu \neq 0$, the convergence being uniform on compact subsets of $\mathbb{R}\backslash\{0\}$. Hence $C_d R_{\lambda+\mathrm{i}\varepsilon}^d E_d C_d$ converges in norm as $\varepsilon \to +0$ for all $\lambda \notin \Theta_d$, uniformly on compact subsets of $\mathbb{R}\backslash\Theta_d$. The compactness of $C_d R_z^d E_d C_d$ is obtained as in lemma 16.14 of AJS.

Remark 6. It is clear from the proof of part (i) and Kato (1966) that $\varphi(\boldsymbol{Q}_c)(H_0 - z)^{-1}\psi(\boldsymbol{Q}_d) \in \mathcal{B}_\infty(\mathcal{H})$ for all $z \in \mathbb{C}$ and converges in norm, uniformly in $\lambda \in \mathbb{R}$, as $z \to \lambda + \mathrm{i}0$, if $\varphi, \psi \in L^{3+\varepsilon}(\mathbb{R}^3) \cap L^{3-\varepsilon}(\mathbb{R}^3)$ and $c \neq d (\varepsilon > 0)$.

Lemma 7. Assume (S1) and (S2). Then $B_a(I - E_a)R_z^a A_a$ converges in operator norm as $z \to \lambda + \mathrm{i}0$, uniformly for $\lambda \in \mathbb{R}$.

Proof. The proof is the same as that of lemma 16.16 in AJS except for minor adjustments due to our weaker hypotheses on the pair potentials. If $2\tau < 0$ is the eigenvalue of $h_a \equiv \gamma_a \boldsymbol{P}_a^2 + V_a$ nearest to 0 and if $\mu \geq -\tau$, then

$$\begin{aligned}&b_a E_{ac}(h_a)(h_a - \mu - \mathrm{i}\eta)^{-1}a_a \\ &\quad = b_a(h_a - \mu - \mathrm{i}\eta)^{-1}a_a - \sum_{k=1}^n (\lambda_k - \mu - \mathrm{i}\eta)^{-1} b_a E_{\{\lambda_k\}} a_a\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of h_a and $E\{\lambda_k\}$ are the associated eigenprojections. The second term has boundary value because $\lambda_k - \mu < -2\tau - \mu < 0$ and because

$$\ell_a E_{\{\lambda_k\}} a_a = \ell_a (|\boldsymbol{P}_a|+1)^{-1}(|\boldsymbol{P}_a|+1)(|h_a|+1)^{-1/2} \times (|\lambda_k|+1)E_{\{\lambda_k\}}(|h_a|+1)^{-1/2}(|\boldsymbol{P}_a|+1)(|\boldsymbol{P}_a|+1)^{-1} a_a$$

is in $\mathscr{B}(L^2(\mathbb{R}^3))$ by lemma 2. The first term is $\ell_a(h_a-\mu-\mathrm{i}\eta)^{-1}a_a = I-(I+\omega^a_{\mu+\mathrm{i}\eta})^{-1}$, which has boundary value for all $\mu \geqslant -\tau$ because of (S1) and (S2). In addition, using the facts that $\|\omega^a_{\mu+\mathrm{i}0}\| \to 0$ as $\mu \to \infty$ (Simon 1971, theorem I.23) and that $\omega^a_{\mu+\mathrm{i}\eta}$ converges to $\omega^a_{\mu+\mathrm{i}0}$ in norm as $\eta \to +0$, uniformly for $\mu \in \mathbb{R}$ (see the proof of lemma 6 or Thomas (1975)), one can show that $\ell^a(h_a-\mu-\mathrm{i}\eta)^{-1}a_a$ converges as $\eta \to +0$ uniformly for $\mu \geqslant -\tau$.

Proof of lemma 3. In view of the definition $\hat{\mathscr{W}}_z := (\hat{I}-\hat{N})(\hat{D}_z-\hat{N})$, it suffices to prove the compactness for $\operatorname{Im} z \neq 0$ of $D_{z,00}$, $D_{z,01}$ and $D_{z,11}$ and the convergence of these operators as $z \to \lambda + \mathrm{i}0$. We first consider an element of $D_{z,00}(c \neq d)$

$$\begin{aligned} B_c(I-E_c)R^c_zA_d &= B_c(I-E_c)(R^0_z - R^c_zV_cR^0_z)A_d \\ &= [I - B_c(I-E_c)R^c_zA_c]B_cR^0_zA_d - [B_cE_c\rho_c^{-2}]\rho_c^2R^0_zA_d. \end{aligned} \tag{A16}$$

By lemma 6 (i) and remark 6, both $B_cR^0_zA_d$ and $\rho_c^2R^0_zA_d$ are compact and have norm boundary values as $z \to \lambda + 0$, uniformly in λ, since $\rho(\boldsymbol{x})^2 \in L^{3+\epsilon}(\mathbb{R}^3) \cap L^{3-\epsilon}(\mathbb{R}^3)$. Next we note that, by (S2) and (S3)

$$\begin{aligned} \|B_cE_c\rho_c^{-2}\| &\leqslant \|B_cE_c\|\|E_c\rho_c^{-2}\| \\ &\leqslant \sum_{\alpha \div c} \|B_cE_c\|\|(I+|\boldsymbol{Q}_c|)^{1+\delta/2}e_\alpha\|_{L^2(\mathbb{R}^3)} < \infty. \end{aligned}$$

Since by lemma 7, $B_c(I-E_c)R^c_zA_c \in \mathscr{B}(\mathscr{H})$ and has norm boundary value as $z \to \lambda + \mathrm{i}0$, uniformly in λ, it follows from (A16) that $D_{z,00} \in \mathscr{B}_\infty(\mathscr{H})$ and has norm boundary value as $z \to \lambda + \mathrm{i}0$, uniformly in λ.

Next, by (15),

$$\begin{aligned} D^{cd}_{z,01} &= (D_{z,00}G_z)^{cd} \\ &= [I - B_c(I-E_c)R^c_zA_c]B_cR^0_zV_dR^d_zE_dC_d - [B_cE_c\rho_c^{-2}]\rho_c^2R^0_zV_dR^d_zE_dC_d \end{aligned}$$

so that the compactness of $D_{z,01}$ follows from that of $D_{z,00}$ and it suffices to prove the existence of the boundary values of $B_cR^0_zV_dR^d_zE_dC_d$ and $\rho_c^2R^0_zV_dR^d_zE_dC_d$. Now by the second resolvent equation

$$\begin{aligned} B_cR^0_zV_dR^d_zE_dC_d &= B_cR^0_zE_dC_d - B_cR^d_zE_dC_d \\ &= [B_cR^0_z\rho_d^2]\rho_d^{-2}E_dC_d - [B_cE_dC_d^{-1}]C_dR^d_zE_dC_d. \end{aligned} \tag{A17}$$

$\rho_d^{-2}E_d$ and $B_cE_dC_d^{-1}$ are bounded operators by virtue of (S3) and lemma 2 respectively. While the norm boundary value for $B_cR^0_z\rho_d^2$ exists as $z \to \lambda + \mathrm{i}0$ for all λ by remark 6, that of $C_dR^d_zE_dC_d$ exists only for $\lambda \in \mathbb{R}\backslash\Theta_d$ (lemma 6). Since $\rho(\boldsymbol{x})^2 \in L^{3+\epsilon}(\mathbb{R}^3) \cap L^{3-\epsilon}(\mathbb{R}^3)$, the same considerations apply to the term $\rho_c^2R^0_zV_dR^d_zE_dC_d$. Combining these observations with (A17), we conclude that $D_{z,01}$ has norm boundary value as $z \to \lambda + \mathrm{i}0$, for all $\lambda \in \mathbb{R}\backslash\Theta$, and the convergence is uniform on each compact subset of $\mathbb{R}\backslash\Theta$.

Finally, $D^{cd}_{z,11} = C_c^{-1}E_cV_dE_dR^d_zC_d = \{C_c^{-1}E_cA_dC_d^{-1}\}\{B_dE_d\}\{C_dR^d_zE_dC_d\}$. Since by lemma 2, the first two factors are bounded, the compactness and existence of boundary values of $D_{z,11}$ follow from lemma 6 (ii).

Proof of lemma 4. (i) Definition (24) implies that $M^\alpha(\varphi_m, \lambda)$ is an integral operator with kernel

$$M^\alpha(\varphi_m, \lambda)(\boldsymbol{\omega}; \boldsymbol{x}_a, \boldsymbol{y}_a) = (2\pi)^{-3/2}\sqrt{\mu_\alpha k_\alpha}\overline{e_\alpha}(\boldsymbol{x}_a)\varphi_m(\boldsymbol{x}_a, \boldsymbol{y}_a)\exp(-\mathrm{i}k_\alpha\boldsymbol{\omega}\cdot\boldsymbol{y}_a). \tag{A18}$$

Thus

$$\|M^\alpha(\varphi_m, \lambda)\|^2_{\mathrm{HS}} = (2\pi)^{-3}\mu_\alpha k_\alpha \int \mathrm{d}\omega \iint \mathrm{d}^3x_a\, \mathrm{d}^3y_a\, |e_\alpha(\boldsymbol{x}_a)|^2|\varphi_m(\boldsymbol{x}_a, \boldsymbol{y}_a)|^2$$

$$\leqslant (2\pi^2)^{-1}\mu_\alpha k_\alpha J \iint \mathrm{d}^3x_a\, \mathrm{d}^3x_b\, |e_\alpha(\boldsymbol{x}_a)|^2\rho(\boldsymbol{x}_a)^{-2m}|\psi(\boldsymbol{x}_b)|^2$$

where J is the Jacobian of transformation from $(\boldsymbol{x}_a, \boldsymbol{y}_a)$ to $(\boldsymbol{x}_a, \boldsymbol{x}_b)$. Since $\psi \in L^2(\mathbb{R}^3)$ and since $\rho(\boldsymbol{x}_a)^{-m}e_\alpha(\boldsymbol{x}_a) \in L^2(\mathbb{R}^3)$ by (S3), we have that $\|M^\alpha(\varphi_m, \lambda)\|_{\mathrm{HS}} \leqslant$ constant $k_\alpha^{1/2}\|\psi\|_2 \cdot \|\rho(\boldsymbol{x})^{-m}e_\alpha\|_2$. The continuity of $M^\alpha(\varphi_m, \lambda)$ in $\mathscr{B}_2$ norm follows from the above estimate and an application of the Lebesgue dominated convergence theorem.

(ii) Since $M^\alpha(A_b, \lambda) = M^\alpha(C_a^2, \lambda)C_a^{-2}E_aA_b$ and $C_a^{-2}E_aA_b \in \mathscr{B}(\mathscr{H})$ by lemma 2 (ii), it suffices to prove the stated results for $M^\alpha(C_a^2, \lambda)$. Since C_a is multiplication by a function of $\boldsymbol{y}_a$ only, we see from (A18) that $M^\alpha(C_a^2, \lambda)$ has the form

$$M^\alpha(C_a^2, \lambda) = M^\alpha(C_a^2, \lambda)E_\alpha = \sqrt{2\mu_\alpha}M_{\rho^2}(k_\alpha^2)E_\alpha \tag{A19}$$

where $M_\varphi(\mu)$: $\mathscr{G}_\alpha \equiv L^2_{x_a}(\mathbb{R}^3) \to L^2(S^2)$ is the two-body operator as defined in (10.5) and (10.6) of AJS. Now $\rho^2(\cdot) \in L^3(\mathbb{R}^3)$, and it is shown in lemma 5 of Kato (1971) that $M_\varphi(\mu) \in \mathscr{B}_4(L^2(\mathbb{R}^3), L^2(S^2))$, depends on μ continuously in $\mathscr{B}_4$ norm if $\varphi(\cdot) \in L^3(\mathbb{R}^3)$, and $\|M_\varphi(\mu)\| \leqslant \|M_\varphi(\mu)\|_4 \leqslant \text{constant}\|\varphi\|_3$.

(iii) By definitions (22)–(25) we find for $f \in \mathscr{S}(\mathbb{R}^6)$ that

$$\|M_a^0(\psi_a, \lambda)f\|^2 = \int_{|\boldsymbol{q}|\leqslant 1} \mathrm{d}^3q\, \mathrm{d}\omega\, |(U_a^0\psi_a f)_\lambda(\boldsymbol{\omega}, \boldsymbol{q})|^2$$

$$= \int_{|\boldsymbol{q}|\leqslant 1} \mathrm{d}^3q\, \mathrm{d}\omega\, h_a(\lambda, q)^2|(\widetilde{\psi_a f})(\gamma_a^{-1/2}\lambda^{1/2}(1-q^2)^{1/2}\boldsymbol{\omega}, (\lambda/\eta_a)^{1/2}\boldsymbol{q})|^2$$

$$= \int_{|\boldsymbol{q}|\leqslant 1} \mathrm{d}^3q\, [2\gamma_a^{1/2}h_a(\lambda, q)^2\lambda^{-1/2}(1-q^2)^{-1/2}]$$

$$\times \|M_\psi(\gamma_a^{-1}\lambda(1-q^2))\hat{f}(\cdot, (\lambda/\eta_a)^{1/2}\boldsymbol{q})\|^2$$

where $M_\psi(\mu)$ is the two-body operator mentioned above which acts in the first variable of $\hat{f}$, defined as the Fourier transform of f with respect to its second variable. Since $\|M_\psi(\mu)\| \leqslant \text{constant}\|\psi\|_3$, we conclude that

$$\|M_a^0(\psi_a, \lambda)f\|^2 \leqslant \text{constant}\|\psi\|_3^2 \int_{|\boldsymbol{q}|\leqslant 1} \mathrm{d}^3q\, \lambda^{3/2}q^2\|\hat{f}(\cdot, (\lambda/\eta_a)^{1/2}\boldsymbol{q})\|^2$$

$$\leqslant \text{constant}\|\psi\|_3^2 \int \mathrm{d}^3k\, \|\hat{f}(\cdot, \boldsymbol{k})\|^2$$

$$= \text{constant}\|\psi\|_3^2\|f\|^2$$

which shows that $\|M_a^0(\psi_a, \lambda)\| \leqslant \text{constant}\|\psi\|_3$. The strong continuity of $M_a^0(\psi_a, \lambda)$ follows from the above expressions, an application of the Lebesgue dominated convergence theorem and the fact that, if $L_nf \to Lf$ for all f in a dense set and if $\|L_n\| \leqslant \kappa$ for all n, then $L_n \to L$ strongly.

Property (iv) of M^α and M^0 follows exactly as in the two-body case (see e.g. lemma 10.3 of AJS and Kato (1971)).

Remark 7. Let $|\varphi_m(\boldsymbol{x}_a, \boldsymbol{x}_b)| \leq \rho(\boldsymbol{x}_a)^{-m}\rho(\boldsymbol{x}_b)$ with $m = 0, 1, 2$ or 3 and $\alpha \div a \neq b$. In such a case, $M^\alpha(\varphi_m, \lambda)$ is in $\mathscr{B}_\infty(L^2(\mathbb{R}^6), L^2(S^{(2)}))$ for $\lambda \in (\lambda_\alpha, \infty)$ and depends on λ continuously in operator norm. To see this we note that, as in (A19),

$$M^\alpha(\varphi_m, \lambda) = M^\alpha(C_a, \lambda)C_a^{-1}E_\alpha\varphi_m = \sqrt{2\mu_\alpha}M_\rho(k_\alpha^2)[E_\alpha C_a^{-1}\varphi_m].$$

We have $E_\alpha C_a^{-1}\varphi_m \in \mathscr{B}(\mathscr{H})$ by (A10) and (S3), whereas the two-body operator $M_\rho(\mu)$ is in $\mathscr{B}_\infty(L^2(\mathbb{R}^3), L^2(S^2))$ for $\mu \in (0, \infty)$ and is norm continuous in μ by lemma 10.21 of AJS.

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